# ON CONTINUOUS CURVES IN $n$ DIMENSIONS*

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If $M_1$ and $M_2$ are subsets of a connected point set $M$, the subset $K$ of $M$ is said to *separate* $M_1$ and $M_2$ in $M$ if $M-K$ is the sum of two mutually separated sets containing $M_1$ and $M_2$ respectively. R. L. Moore‡ has shown that in order that a plane continuum $M$ be a continuous curve§ it is necessary and sufficient that for every two distinct points $A$ and $B$ of $M$ there should exist a subset of $M$ which consists of a finite number of continua and which separates $A$ and $B$ in $M$. Consider the following example: Let $S_i$ $(i=1, 2)$ be the set of all points $(x, y, z)$ in three dimensions such that $x=(-1)^i$, $-1\leqq y\leqq 1$, $0\leqq z\leqq 1$. Let $R_0$ be the set of all points $(x, y, z)$ such that $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, $z=0$. For each integer $n>0$, let $R_n$ be the set of all points $(x, y, z)$ such that $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, $z=1/n$. Let

$$M = S_1 + S_2 + \sum_{n=0}^{\infty} R_n.$$

It is easy to see that every two points of $M$ may be separated by a single subcontinuum of $M$ and yet $M$ is not a continuous curve. Hence the condition given by Moore is not sufficient in order that a continuum in $n$ dimensions $(n>2)$ be a continuous curve. In this paper we give two modifications (Theorems 2 and 4) of Moore's theorem which hold in $n$ dimensions.

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‡ *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–307.

§ We shall use the term *continuous curve* in the sense of a point set which is closed, connected and connected im kleinen. See R. L. Moore, *Concerning simple continuous curves*, Transactions of this Society, vol. 21 (1920), p. 347.

THEOREM 1.* *If $M$ is a continuous curve in euclidean space of $n$ dimensions, $K$ is a bounded subcontinuum of $M$ and $\epsilon$ is any positive number, then there exists a set $L$ such that* (1) *$K+L$ is a continuous curve which is a subset of $M$,* (2) *every point of $L$ is within a distance $\epsilon$ of some point of $K$,* (3) *$L$ consists of a countable set of arcs of $M$, not more than a finite number of which are of diameter greater than any given positive number,* (4) *$L+K$ is non-dense at every point except those points at which $K$ fails to be non-dense.*

PROOF. The set $M$ is uniformly connected im kleinen over the set $K$.† Let $\delta_1, \delta_2, \delta_3, \cdots$ be a sequence of positive numbers such that every two points of $K$ whose distance from one another is less than $\delta_m$ can be joined by an arc of $M$ whose diameter is less than $\epsilon/2m$. For each point $p$ of $K$ and each positive integer $n$, let $C_{np}$ and $C'_{np}$ be hyperspheres with center $p$ and radii $\epsilon/n$ and $\epsilon/(2n)$ respectively.‡ By the Borel theorem, for each value of $n$ there is a finite subset of the set $[C'_{np}]$,

$$C'_{np_{n1}}, C'_{np_{n2}}, C'_{np_{n3}}, \cdots, C'_{np_{nn'}},$$

such that every point of $K$ is in the interior of one of the sets $C'_{np_{ni}}$ for $1 \leqq i \leqq n'$. Since $M$ is a continuous curve there are but a finite number,

$$M_{ni1}, M_{ni2}, M_{ni3}, \cdots, M_{nim_n},$$

of the components§ of $M \cdot I(C_{np_{ni}})$ that contain points in the interior of $C'_{np_{ni}}$. For each $n$, $i$ and $j$, let $[K_{nij}]$ be the set of

* This theorem contains as a special case a theorem due to H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27 (1925), pp. 29–46, Theorem 3.

† S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), p. 173.

‡ If $p$ is a point and $r$ a positive number, the *hypersphere* with center $p$ and radius $r$ is the set of all points of the space whose distance from the point $p$ is $r$. If $S$ is a hypersphere, $I(S)$ denotes the interior of $S$.

§ A connected subset of a point set $H$ which is not a *proper* subset of any connected subset of $H$ is called a *component* of $H$.

components of $K\cdot M_{nij}\cdot I(C'_{np_{ni}})$. By the Zermelo postulate*, there exists a set of points $[P_{nij}]$ such that each set $K_{nij}$ contains just one point $P_{nij}$ and each point $P_{nij}$ belongs to just one component $K_{nij}$. In the set $[P_{nij}]$ there is a finite subset,

$$\overset{1}{P}_{nij},\ \overset{2}{P}_{nij},\ \overset{3}{P}_{nij},\ \cdots,\ \overset{k_1}{P}_{nij},\dagger$$

such that every point of $[P_{nij}]$ is within a distance $\delta_1$ of some point of this finite set. There exists an arc $\overset{r}{\alpha}_{nij}$ $(1\leqq r\leqq k_1-1)$ with end points $\overset{r}{P}_{nij}$ and $\overset{r+1}{P}_{nij}$ and lying wholly in $M_{nij}$. There exists a finite subset,

$$\overset{k_1+1}{P}_{nij},\ \overset{k_1+2}{P}_{nij},\ \cdots,\ \overset{k_2}{P}_{nij},$$

of the set $[P_{nij}]$ such that every point of $[P_{nij}]$ is within a distance $\delta_2$ of some point of $\overset{1}{P}_{nij}$, $\overset{2}{P}_{nij}$, $\cdots$, $\overset{k_2}{P}_{nij}$. Let $\overset{r}{\alpha}_{nij}$ $(k_1\leqq r\leqq k_2-1)$ be an arc of $M_{nij}$ with end points $\overset{r+1}{P}_{nij}$ and some point of $\sum_{s=1}^{k_1}\overset{s}{P}_{nij}$. Continue this process indefinitely except that for $t>n$ we place the additional condition on $\overset{r}{\alpha}_{nij}(k_t\leqq r\leqq k_{t+1}-1)$ that it be of diameter less than $\epsilon/(2t)$. This can be done since any two points of $K$ whose distance from one another is less than $\delta_t$ can be joined by an arc of $M$ whose diameter is less than $\epsilon/(2t)$.

For each $n, i$ and $j$, there is a countable set of arcs of $M$, $\overset{1}{\alpha}_{nij}$, $\overset{2}{\alpha}_{nij}$, $\overset{3}{\alpha}_{nij}$, $\cdots$, such that (a) each lies interior to a hypersphere of radius $\epsilon/n$ with a point of $K$ as center, (b) only a finite number are of diameter greater than a given positive number, and (c) each has its end points on $K$. For each value of $n$ the numbers $i$ and $j$ range over finite sets of values; hence the set of all arcs $[\overset{r}{\alpha}_{nij}]$ for a fixed value of $n$ satisfy conditions (a), (b), and (c) above. And since all arcs $[\overset{r}{\alpha}_{nij}]$ for a fixed value of $n$ are of diameter less than $2\epsilon/n$, the set of all arcs $[\overset{r}{\alpha}_{nij}]$ for all values of $n$ satisfies the condi-

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* E. Zermelo, *Untersuchung über die Grundlagen der Mengenlehre*, Mathematische Annalen, vol. 65 (1908), pp. 261–281.

† The symbol $k_1$ denotes a positive integer whose value depends on $n$, $i$ and $j$.

tion that only a finite number are of diameter greater than a given positive number. Let

$$L = \sum \alpha_{nij}^{r}, \quad \begin{matrix} 1 \leqq i \leqq n', & 1 \leqq j \leqq m_n, \\ 1 \leqq r < \infty, & 1 \leqq n < \infty. \end{matrix}$$

We have shown that $L$ satisfies conditions (2) and (3) of our theorem. It remains to prove that (1) and (4) are satisfied. Since only a finite number of the arcs of $L$ are of diameter greater than a given positive number and each has a point on the closed set $K$, every limit point of $L$ which does not belong to $L$ belongs to $K$. Thus $K+L$ is closed. Let $P$ be any point of $K+L$. If $P$ does not belong to $K$ it is easy to see that $K+L$ is connected im kleinen at $P$, for the interiors of hyperspheres of sufficiently small radii and center $P$ contain no point of $K$ and points of only a finite number of arcs of $L$. If $P$ is a point of $K$ and $\eta$ is any positive number, there is a hypersphere $C_{np_{ni}}$ which lies entirely in the interior of the hypersphere with radius $\eta/4$ and center $P$ and such that $I(C'_{np_{ni}})$ contains $P$. Let $M_{nij}$ be the component of $M \cdot I(C_{np_{ni}})$ containing $P$. There exists a positive number $\gamma$ such that every point of $K$ whose distance from $P$ is less than $\gamma$ lies in $M_{nij}$. There exists a number $\rho > 0$ such that every point $p'$ of $L$ whose distance from $P$ is less than $\rho$ lies on an arc $\alpha_{p'}$ of $L$, one of whose points $e$ belongs to $K \cdot M_{nij}$ and such that the subarc $p'e$ of $\alpha_{p'}$ is of diameter less than $\eta/2$.* Let $\sigma$ be the smaller of $\gamma$ and $\rho$. Now let $Q$ be any point of $K+L$ whose distance from $P$ is less than $\sigma$. If $Q$ belongs to $K$ it belongs to $M_{nij}$. By the method of

* If $S_\gamma$ and $S_d$ denote hyperspheres with center $P$ and radii $\gamma$ and $d$ respectively, then only a finite number of arcs of $L$ have points in $I(S_d)$ for any $d < \gamma$ and contain no point of $I(S_\gamma) \cdot K$ since any such arc is at least of diameter $\gamma - d$. There is a number $d_1 > 0$ such that for $d \leqq d_1$ there is no such arc. Also there is a number $d_2 > 0$ such that no arc of $L$ of diameter greater than $\eta/2$ contains a point whose distance from $P$ is less than $d_2$ unless the arc contains $P$. On each of the finite set of arcs of $L$ of diameter greater than $\eta/2$ that contain $P$ there is a point $q$ such that the subarc $qP$ of the arc is of diameter less than $\eta/2$. Let $d_3$ be the smallest of the finite set of distances from $P$ to the points $q$. Let $\rho$ be the smallest of the numbers $d_1$, $d_2$ and $d_3$.

construction of $L$, there is a subset $L'$ of $L$ such that $M_{nij}$ contains $L'$ and $L'+K\cdot M_{nij}$ is connected. But every point of $L'+K\cdot M_{nij}$ is at a distance from $P$ less than $\eta/2$ and $L'+K\cdot M_{nij}$ contains both $P$ and $Q$. If $Q$ is not a point of $K$, it lies on an arc $\alpha_Q$ of $L$ which contains a point $e$ of $K\cdot M_{nij}$ such that the subarc $eQ$ of $\alpha_Q$ is of diameter less than $\eta/2$. Then $\alpha_Q+L'+K\cdot M_{nij}$ is a connected subset of $L+K$ containing $P$ and $Q$ and such that every point is at a distance from $P$ less than $\eta$. Therefore $K+L$ is connected im kleinen at every point $P$.

Let $P$ be any point of $K$ at which $K$ is non-dense. Then if $S_1$ is any hypersphere with center $P$, the set $I(S_1)$ contains a hypersphere $S_2$ such that $S_2+I(S_2)$ contains no point of $K$. Since only a finite number of the arcs of $L$ are of diameter greater than a given positive number, there are only a finite number of arcs of $L$ that have points in $I(S_2)$. Then there is a hypersphere $S_3$ lying in $I(S_2)$ such that $I(S_3)$ contains no point of $L$. Then the interior of every hypersphere $S_1$ with center at $P$ contains a hypersphere $S_3$ such that $I(S_3)$ contains no point of $K+L$. Hence $K+L$ is non-dense at the point $P$.

THEOREM 2. *In order that a continuum $M$ lying in euclidean space of $n$ dimensions be a continuous curve it is necessary and sufficient that for every two distinct points $A$ and $B$ of $M$ there should exist a subset of $M$ which consists of a finite number of continuous curves and which separates $A$ and $B$ in $M$.*

PROOF. The condition is necessary. Let $d$ be the distance from $A$ to $B$. Let $S_1$ and $S_2$ be hyperspheres with center $A$ and radii $d/2$ and $d/4$ respectively. Let $H=S_1+I(S_1)-I(S_2)$. The set $M\cdot H$ is closed and it is easy to see that there is at least one component of $M\cdot H$ containing points on both $S_1$ and $S_2$. As $M$ is a continuous curve there cannot be more than a finite number of such components. Let $K_1$, $K_2$, $K_3, \cdots, K_m$ denote the set of all components of $M\cdot H$ which contain a point on $S_1$ and a point on $S_2$. By Theorem 1, for each $i$, $1\leqq i\leqq m$, there is a continuous curve $M_i$ which contains

$K_i$, is a subset of $M$ and such that every point of $M_i$ is within a distance $d/8$ of some point of $K_i$. Suppose that $A$ and $B$ lie in a connected subset of $M-\sum_{i=1}^{m} M_i$. Then there is an arc with end points $A$ and $B$ lying in $M-\sum_{i=1}^{m} M_i$.* This arc contains a subarc $\alpha$ which is a subset of $H$ and has one end point on $S_1$ and the other on $S_2$. Then $\alpha$ must belong to some set $K_i$ and thus to $\sum_{i=1}^{m} M_i$. But this is impossible, for $M-\sum_{i=1}^{m} M_i$ contains $\alpha$. Therefore $\sum_{i=1}^{m} M_i$ separates $A$ and $B$ in $M$.

The condition is sufficient. If $M$ is not a continuous curve there exist two concentric hyperspheres $S_1$ and $S_2$ and an infinite set of subcontinua $M_\infty$, $M_1$, $M_2$, $M_3$, $\cdots$ of $M$ satisfying the conditions of the Moore-Wilder lemma.† Let $S_3$ and $S_4$ be distinct hyperspheres concentric with $S_1$ and lying between $S_1$ and $S_2$. Each continuum $M_i$ contains a subcontinuum $K_i$ which contains a point $P_i$ on $S_3$ and a point $Q_i$ on $S_4$ and is a subset of the set $G$ consisting of $S_3$ and $S_4$ and all points which lie between $S_3$ and $S_4$. There exists a sequence of integers $n_1$, $n_2$, $\cdots$, such that $[P_{n_i}]$ has a sequential limit point $A$ and $[Q_{n_i}]$ has a sequential limit point $B$. By hypothesis there exists a finite set of continuous curves $C_1$, $C_2$, $C_3$, $\cdots$, $C_m$ which are subsets of $M$ and separate $A$ and $B$ in $M$.

CASE I. Suppose infinitely many of the continua $K_{n_i}$ contain a point of $\sum_{k=1}^{m} C_k$. As there are but a finite number of the curves $C_k$, one curve $C_{k'}$ must contain a point $p_{n_i}$ of $K_{n_i}$ for infinitely many values of $i$. The set $[p_{n_i}]$ has a limit point $P$, which must belong to $M_\infty$ and to $G$. Let $\epsilon$ be a positive number such that no point of $S_1+S_2$ is within a distance $\epsilon$ of $P$. As $C_{k'}$ is a continuous curve, the point $P$

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* R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260. Moore's theorem is stated for two dimensions, but the extension to $n$ dimensions is obvious.

† R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), p. 296; R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 371. The lemma holds equally well for $n$ dimensions and for unbounded continua.

belongs to $C_{k'}$ and there is a number $\delta_\epsilon > 0$ such that any point of $C_{k'}$ whose distance from $P$ is less than $\delta_\epsilon$ can be joined to $P$ by an arc of $C_{k'}$ of diameter less than $\epsilon$. There is a point $p_{n_r}$ of $[p_{n_i}]$ whose distance from $P$ is less than $\delta_\epsilon$. Let $\alpha$ denote an arc of $C_{k'}$ with end points $P$ and $p_{n_r}$ and of diameter less than $\epsilon$. The arc $\alpha$ contains a point of $M_{n_r}$ and a point of $M_\infty$ and lies entirely between $S_1$ and $S_2$. By the Moore-Wilder lemma, $M_{n_r}$ is a component of the common part of $M$ and the set composed of $S_1$ and $S_2$ and all points lying between $S_1$ and $S_2$. Hence $M_{n_r}$ contains the arc $\alpha$. But this contradicts the condition of the lemma that $M_{n_r}$ and $M_\infty$ have no common points.

CASE II. Suppose only a finite number of the continua $K_{n_i}$ contain points of $\sum_{k=1}^{m} C_k$. The set $M - \sum_1^m C_k$ is the sum of two mutually separated sets $M_A$ and $M_B$ containing $A$ and $B$ respectively. Every set $K_{n_i}$ which contains no point of $\sum_1^m C_k$ lies wholly in $M_A$ or wholly in $M_B$. There is an integer $j$ such that for $i \geqq j$, the continuum $K_{n_i}$ contains no point of $\sum_1^m C_k$. Both $A$ and $B$ are limit points of the set $\sum_{i=j}^{\infty} K_{n_i}$. Either infinitely many of the sets $K_{n_i}$ $(i \geqq j)$ belong to $M_A$ or infinitely many belong to $M_B$. If the first holds then $B$ is a limit point of $M_A$; under the second possibility the point $A$ is a limit point of the set $M_B$. In either possibility we have a contradiction since $M_A$ and $M_B$ are mutually separated.

The assumption that $M$ is not a continuous curve leads to a contradiction with the assumed condition in either case. Therefore the condition is sufficient.

It is to be noticed that in the proof of the necessity of the condition in Theorem 2 we showed that the separating continuous curves were bounded. Hence we have the following corollary and theorem.

COROLLARY. *If $A$ and $B$ are points of a continuous curve $M$ lying in euclidean space of $n$ dimensions, there exists a subset of $M$ which consists of a finite number of bounded continuous curves and which separates $A$ and $B$ in $M$.*

THEOREM 3. *If $K_1$ and $K_2$ are any two mutually exclusive and closed point sets, one of which is bounded, then $K_1$ lies wholly in a finite number of the complementary domains of $K_2$.*

PROOF. Suppose the contrary is true. Then there exists an infinite sequence $D_1, D_2, D_3, \cdots$ of distinct complementary domains of $K_2$ each of which contains at least one point of $K_1$. For each positive integer $i$, let $P_i$ denote a point of $K_1$ belonging to $D_i$. Let $H$ denote the set of points $P_1+P_2+P_3+\cdots$. By hypothesis either $K_1$ or $K_2$ is bounded. If $K_1$ is bounded, then $H$ is bounded because $H$ is a subset of $K_1$; and if $K_2$ is bounded, then since $H$ contains at most one point in the unbounded complementary domain of $K_2$, it readily follows that $H$ is bounded. Hence, in any case, $H$ is bounded; and since it is infinite, it must have at least one limit point $P$. Since $K_1$ is closed and contains $H$, it must contain the point $P$; and since $K_1$ and $K_2$ are mutually exclusive, $P$ must belong to some complementary domain $D$ of $K_2$. Clearly this is impossible, since $P$ is a limit point of $H$, and not more than one point of $H$ can belong to $D$. Thus the supposition that Theorem 3 is not true leads to a contradiction.

THEOREM 4. *In order that a continuum $M$ in a euclidean space $E_n$ of $n$ dimensions should be a continuous curve it is necessary and sufficient that every two mutually exclusive, closed, and bounded subsets of $M$ should be separated in $M$ by the sum of a finite number of subcontinua of $M$.*

PROOF.* The condition is sufficient. For suppose a continuum $M$ satisfies the condition but is not a continuous curve. Then by the Moore-Wilder lemma† it follows that there exist two different concentric hyperspheres $C_1$ and $C_2$ and a countable infinity of mutually exclusive subcontinua of $M$: $W, M_1, M_2, M_3, \cdots$ such that (1) if $D$ denotes the

* Compare this proof with that given by R. L. Moore for Theorem 1 of his paper, *A characterization of a continuous curve*, loc. cit.

† See reference to the Moore-Wilder lemma above.

$n$-dimensional domain whose boundary is $C_1+C_2$, then each of these continua contains at least one point on each of the hyperspheres $C_1$ and $C_2$, and each of them, save possibly $W$, is a component of the set of points $M\cdot(D+C_1+C_2)$, and (2) $W$ is the sequential limiting set of the sequence of continua $M_1, M_2, M_3, \cdots$. Let $A$ and $B$ denote the sets of points $W\cdot C_1$ and $W\cdot C_2$ respectively and, for each positive integer $i$, let $a_i$ denote the set of points $M_i\cdot C_1$ and $b_i$ the set $M_i\cdot C_2$. Since $A$ and $B$ are mutually exclusive, closed, and bounded subsets of $M$, by hypothesis there exists a subset $L$ of $M$ such that (1) $M-L$ is the sum of two mutually separated point sets $M_a$ and $M_b$ containing $A$ and $B$ respectively, and (2) $L$ is the sum of a finite number of continua $L_1, L_2, L_3, \cdots, L_m$. Since neither $A$ nor $B$ has a point in common with $L$, and $A$ contains no point of $M_b$ and $B$ contains no point of $M_a$, therefore there exist open sets $C_a$ and $C_b$, containing $A$ and $B$ respectively, such that $C_a$ contains no point of $L+M_b$ and $C_b$ contains no point of $L+M_a$. There exists an integer $\delta$ such that, for every $j$ greater than $\delta$, the point set $a_j$ lies wholly in $C_a$ and the point set $b_j$ lies wholly in $C_b$. Thus, for every $j$ greater than $\delta$, $M_j$ contains a point of $M_a$ and also a point of $M_b$. But $M_j$ is a subcontinuum of $M$, and every subcontinuum of $M$ which contains a point of each of the sets $M_a$ and $M_b$ must contain at least one point of $L$. Hence, for every $j$ greater than $\delta$, $M_j$ contains a point of $L$, and therefore of some one of the sets $L_1, L_2, \cdots, L_m$. It follows that there exists an integer $g$ and an infinite sequence of distinct positive integers $t_1, t_2, t_3, \cdots$ such that, for every $j$, $L_g$ contains at least one point in common with $M_{t_j}$. Since, for every $j$, the subcontinuum $L_g$ of $M$ contains a point of $M_{t_j}$ and a point of $M_{t_{j+1}}$ it follows by a lemma of R. L. Moore's* that $L_g$ must contain a point either of $a_{t_j}$ or of $b_{t_j}$. Thus there exists an infinite sequence of distinct integers $j_1, j_2, j_3, \cdots$, such that either $L_g$ has a point in common with each point set

* *A characterization of a continuous curve*, loc. cit., Lemma 2.

of the sequence $a_{j_1}, a_{j_2}, a_{j_3}, \cdots$, or it has at least one point in common with each point set of the sequence $b_{j_1}, b_{j_2}, b_{j_3}, \cdots$. In the first case it readily follows that $A$ contains at least one point of $L_g$, and in the second case that $B$ contains at least one point of $L_g$. But $A+B$ is a subset of $M-L$. Thus the supposition that $M$ is not a continuous curve leads to a contradiction.

The condition is also necessary. For let $M$ be any continuous curve in $E_n$, and let $K_1$ and $K_2$ be any two mutually exclusive, closed, and bounded subsets of $M$. It follows by Theorem 3 that there exists a finite number $D_1, D_2, D_3, \cdots, D_m$ of the complementary domains of $K_2$ whose sum contains the point set $K_1$. For each positive integer $i \leqq m$, let $B_i$ denote the boundary of $D_i$, let $H_i$ be the set of points common to $K_1$ and $D_i$, and let $4d_i$ be the minimum distance between the closed sets of points $H_i$ and $B_i$. For each point $P$ of $H_i+B_i$, let $C_p$ denote a hypersphere with $P$ as center and radius $d_i$, and let $G_i'$ be the collection of all the hyperspheres $[C_p]$ for all points $P$ of $H_i+B_i$. Since $K_1+K_2$, and hence also $H_i+B_i$, is bounded, then by the Borel theorem there exists a finite subcollection $G_i$ of the hyperspheres of $G_i'$ such that $H_i+B_i$ is a subset of the sum $I_i$ of the interiors of the collection $G_i$. Let $T_i$ denote the point set $(D_i+B_i) - I_i \cdot (D_i+B_i)$. Then $T_i$ is closed. Let $F_i$ denote the sum of all the hyperspheres (not including their interiors) of the collection $G_i$ which enclose at least one point of $H_i$, and let $N_i$ be the sum of all those which enclose at least one point of $B_i$. Since the least distance between $H_i$ and $B_i$ is $4d_i$, and since the radius of each hypersphere of $G_i$ is $d_i$, it follows that $F_i$ and $N_i$ are mutually exclusive closed sets whose least distance apart is $>d_i$. Let $Q_i$ denote the collection of all those maximal connected subsets of $M$ which lie wholly in $T_i$ and contain at least one point of each of the sets $F_i$ and $N_i$. Each element of $Q_i$ is a continuum, and since $M$ is a continuous curve, it follows by the Moore-Wilder lemma that $Q_i$ has just a finite number of elements. Hence $Q_i$ is a finite collection of mutually exclusive continua $L_{1i}$, $L_{2i}$, $L_{3i}$, $\cdots$, $L_{n_i i}$ which belong to $M$.

Now let $L$ denote the point set $\sum_{i=1}^{m}\sum_{j=1}^{n_i}L_{ji}$. Then $L$ is the sum of a finite number of mutually exclusive subcontinua of $M$. Let $M_a$ denote the sum of all those components of $M-L$ which contain at least one point of $A$, and let $M_b$ denote the point set $M-(M_a+L)$. No point of $B$ belongs to $M_a$. For if a point $X$ of $B$ belonged to $M_a$, then* $X$ could be joined in $M$ to some point $Y$ of $A$ by an arc which contains no point of $L$, and this arc would contain a subarc $t$ which is a subset of some set $T_i$ and which has its end points on $F_i$ and $N_i$ respectively; and the arc $t$ would necessarily be a subset of some continuum of the collection $Q_i$, contrary to the fact that $t$ contains no point of $L$. Therefore $B$ must be a subset of $M_b$. Since $M$ is connected im kleinen and $L$ is closed, it readily follows that $M_a$ and $M_b$ are mutually separated. Hence $M-L$ is the sum of two mutually separated sets $M_a$ and $M_b$ containing $A$ and $B$ respectively, and therefore $L$ separates $A$ and $B$ in $M$.

THEOREM 5. *In order that a continuum $M$ in a space of $n$ dimensions should be a Menger regular curve*† *it is necessary and sufficient that every two points of $M$ should be separated in $M$ by some finite subset of $M$.*

PROOF. The condition is sufficient. Let $P$ be any point of $M$ and $\epsilon$ any positive number. Let $C_1$ and $C_2$ be two distinct hyperspheres each of which has $P$ as center and is of radius less than $\epsilon/4$. Let $D$ denote the domain between $C_1$ and $C_2$, and let $K$ denote the set of points common to $D+C_1+C_2$ and to $M$. Then $K$ is closed. Now by Theorem 2 it follows

* R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

† A continuum $M$ is said to be a Menger regular curve provided that for each point $P$ of $M$ and each positive number $\epsilon$ there exists an open subset $T$ of $M$ of diameter less than $\epsilon$ which contains $P$ and whose $M$-boundary is finite. The $M$-boundary of an open subset $T$ of a continuum $M$ is the set of all those points of $M-T$ that are limit points of $T$. See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925–1926), pp. 276–306.

that $M$ is a continuous curve. By hypothesis, for each point $X$ of $K$ there exists a finite subset $N_x$ of $M$ which separates $X$ and $P$ in $M$. For each such point $X$, the maximal connected subset $H_x$ of $M-N_x$ which contains $X$ is an open subset of $M$ which does not contain $P$ and whose $M$-boundary is finite (a subset of $N_x$). Let $G_0$ denote the collection of sets $[H_x]$ for all points $X$ of $K$. Since $K$ is closed and bounded, then by the Borel theorem the collection $G_0$ contains a finite subcollection $G$ which covers $K$. Let $R$ denote the sum of all the point sets of the collection $G$. Then $K$ is a subset of $R$, and $R$ is an open subset of $M$. Furthermore $B$, the $M$-boundary of $R$, is finite, for $R$ is the sum of a finite number of the sets $H_x$. Now, supposing that $C_1$ is within $C_2$, let $A$ denote the set of all those points of $B$ which lie on or within $C_1$. Now $R+A$ does not contain $P$, for $P$ belongs to no set $H_x$ and to no $N_x$. Let $T$ denote the maximal connected subset of $M-A$ which contains $P$. It is readily seen that $T$ must lie within $C_1$. Hence the diameter of $T$ is less than $\epsilon$. The $M$-boundary of $T$ is finite, because it is a subset of $A$. Then, since $T$ is an open subset of $M$, it follows that $P$ is a regular point of $M$ and that $M$ is a Menger regular curve.

That the condition is necessary follows at once from the definition of a Menger regular curve.

THEOREM 6. *If every two points of a continuum $M$ are separated in $M$ by some finite subset of $M$, then every two mutually exclusive, closed, and bounded subsets of $M$ are separated in $M$ by some finite subset of $M$.*

PROOF. It follows by Theorem 5 that $M$ is a Menger regular curve. Then by a theorem of Menger's,* it follows that every two mutually exclusive, closed, and bounded subsets of $M$ can be separated in $M$ by some finite subset of $M$.

THE UNIVERSITY OF TEXAS

* Loc. cit., Theorem 12.