THE PROBLEM OF DEPRECIATION IN THE CALCULUS OF VARIATIONS*

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1. *Introduction.* In a recent article Hotelling has shown that the older treatments of depreciation involve a number of serious errors of reasoning and has formulated the problem in such a way that many of these errors are overcome.‡ His view-point is that the owner of a machine will do everything in his power to maximize the present value of the sum of the anticipated rentals which the machine will yield from the present time t_1 to some future time t_2 plus the present value of the salvage value of the machine at the time t_2 when it is salvaged. Although the depreciation problem appears to be a Lagrange problem in the calculus of variations,§ Hotelling has chosen to consider it as a problem in the theory of ordinary maxima of functions.

If, in the light of recent developments in the new dynamical economics,|| the depreciation problem is formulated as a Lagrange problem with variable end-points, the resulting problem is sufficiently general to include as special cases all of the existing depreciation theories, i.e., such theories as the straight line, the compound interest, the sinking fund, the unit-cost-plus, and the theory due to Hotelling. In

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‡ H. Hotelling, *A general mathematical theory of depreciation*, Journal of the American Statistical Association, September, 1925.

§ Bliss, *Lectures on the Calculus of Variations*, University of Chicago, Summer, 1925, mimeographed by O. E. Brown, Northwestern University, Evanston, Ill.

|| Roos, *Dynamical economics*, Proceedings of the National Academy, vol. 13, No. 3 (March, 1927); Roos, *A dynamical theory of economics*, Journal of Political Economy, October, 1927. See also, Roos, *A mathematical theory of competition*, American Journal of Mathematics, vol. 57, No. 3, July, 1925, and G. C. Evans, *Dynamics of monopoly*, American Mathematical Monthly, vol. 31 (1924), No. 2.

as much as existing depreciation theories assume a constant rate of production, the importance of a calculus of variations treatment which allows the rate of production to vary, is immediately evident.

2. *The Value of a Machine.* In order to obtain an expression for the operating expense or cost of production for a machine or other property, mathematical economists, following the example set by Walras, define certain quantities called coefficients of production. These coefficients are defined as the quantities of the services of the factors of production, i.e. services of land, services of persons and services of capital, that enter into the manufacture of a unit of a given commodity.* For the static case Walras assumes these quantities to be constant. For the dynamic case an obvious extension would be to suppose the coefficients to be functions of the time, but this is not enough. There seems to be justification for writing the coefficients of production for a commodity C as functions of the rate of production of C, the selling price of C and the first derivatives of these quantities with respect to time.† In this paper we will, therefore, suppose the coefficients of production to be functions of the rate of production of C, the price of C, and the first derivatives of these quantities with respect to the time.

If there is one producer manufacturing an amount $u(t)$ of C in unit time and if, furthermore, the selling price of one unit of C is $p(t)$, the coefficients of production are functions

$$f_\alpha(u, u', p, p', t), \qquad (\alpha = 1, \cdots, m),$$

where m is the number of services required to manufacture one unit of C and primes denote derivatives with respect to time. If we denote the prices of the m services required to produce one unit of C by $p_\alpha(t)$, then by the definition

* V. Pareto, *Manuel d'Economie Politique*, pp. 607.

† Roos, *Dynamical economics*, loc. cit.

of the coefficients of production the cost of producing $u(t)$ units of C is

$$(1)\quad \psi(u,u',p,p',p_1,\cdots,p_m,t) = \sum_{\alpha=1}^{m} u(t)f_\alpha(u,u',p,p',t)p_\alpha(t).$$

This gives the cost of production before depreciation has been taken into account.

The profit or rent which will be obtained from a machine in unit time will, therefore, be

$$R(t) = p(t)u(t) - \psi(u,u',p,p',p_1,\cdots,p_m,t).$$

Now, the value of a machine to its operator at a time t_1 is the sum of the anticipated rentals which it will yield from the time t_1 to a time ω at which it is to be salvaged each multiplied by a discount factor to allow for interest plus the salvage value also discounted. In the most general case the interest will vary with the time. We can, therefore, for the general case write the value of a machine to its operator at a time t_1 as

$$(2)\quad V = \int_{t_1}^{\omega} [pu - \psi(u,u',p,p',p_1,\cdots,p_m,t)]e^{-\int_{t_1}^{t}\delta(\nu)d\nu}dt + Se^{-\int_{t_1}^{\omega}\delta(\nu)d\nu},$$

where $\delta(\nu)$ is the force of interest, which is defined as the rate of increase of an invested sum s divided by s, and S is the salvage value of the machine at the time ω.* This salvage value S is the cost price K of the machine at the time t_1 minus the depreciation in the market value of the machine after it has been operated for the period of time t_1 to ω. The depreciation in the market value of the machine is in general a functional of the rate of production, of the price of the article produced and of the time derivatives of these quantities. We may, therefore, write

$$(3)\qquad S = K - \int_{t_1}^{t_2} D(u,u',p,p',t)e^{-\int_{t_2}^{t}\delta(\nu)d\nu}\,dt$$

* Hotelling, *A general mathematical theory of depreciation*, loc. cit., p. 342.

where $D(u, u', p, p', t)$ is the rate of depreciation. If we substitute the value of S defined by (3) in equation (2), we obtain on transposing the second term of the right-hand member

$$\left[V - Ke^{-\int_{t_1}^{t_2}\delta(\nu)d\nu}\right] = \int_{t_1}^{t_2}[pu$$

$$- \psi(u,u',p,p',p_1, \cdots, p_m,t) - D(u,u',p,p',t)]e^{-\int_{t_1}^{t}\delta(\nu)d\nu}\, dt.$$

The quantity represented by the second term of the left hand member is the value at t_1 of a sum K necessary to replace the machine at the time t_2. In order to simplify notation in the work which follows let us write this last expression in the form

$$(4) \quad I = \int_{t_1}^{t_2}[pu - Q(u,u',p,p',p_1, \cdots, p_m,t)]e^{-\int_{t_1}^{t}\delta(\nu)d\nu}\, dt,$$

where

$$Q(u, u', p, p', p_1, \cdots, p_m, t)$$
$$= \psi(u, u', p, p', p_1, \cdots, p_m, t) + D(u,u', p, p', t).$$

3. *Equations of Demand and Supply.* For the commodity produced by the machine there will be a function ξ defining the rate of demand. As I have already pointed out, this rate of demand in its most general form will be a functional of the type

$$\xi = g(u,u',p,p',t) + \int_{t_1}^{t} H(u,u',p,p',t,s)ds$$

where as before primes indicate derivatives with respect to time and s is a parameter of integration.*

If as many units are sold as are produced, the demand per unit time will be $u(t)$ so that the demand equation will be

$$(5) \qquad \phi(u,u',p,p',t) = \int_{t_1}^{t} H(u,u',p,p',t,s)ds$$

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* Roos, *A dynamical theory of economics*, loc. cit.

where as notation $\phi(u, u', p, p', t) = u - g(u, u', p, p', t)$. When the rate of demand depends only slightly upon the history of prices and rates of production, we are justified in writing $H \equiv 0$. For this case we obtain a first order differential equation of demand

$$\phi(u,u',p,p',t) = 0 \qquad (t_1 \leqq t \leqq t_2). \tag{6}$$

In as much as this last form of the demand equation is sufficiently general to include as special cases all forms of the demand equation now used in statistical investigations, it will be sufficiently general for the purposes of this paper.

4. *Necessary Conditions for a Solution.* A likely assumption to make regarding the operation of the machine is that the operator will endeavor to maximize the expression (4), which is the difference between the value of the machine at the time t_1 and the discounted cost price. This assumption is not equivalent to the assumption that the operator will endeavour to maximize V, for t_2 is a variable end-value, but it has the advantage of simplicity.* We will find it convenient to introduce the notation $y_1(t) = u(t)$ and $y_2(t) = p(t)$ in this paragraph in stating the problem of depreciation and in writing the conditions necessary for a solution. Under the hypothesis that (4) is to be maximized, we may state the problem of depreciation as that of choosing the rate of production $y_1(t)$ and the price $y_2(t)$ satisfying a differential equation of demand

$$\phi(y_1, y_1', y_2, y_2', t) = 0 \qquad (t_1 \leqq t \leqq t_2),$$

and having end-points which satisfy end-equations

$$\theta_\mu[t_1, y_1(t_1), y_2(t_1), t_2, y_1(t_2), y_2(t_2)] = 0, \quad (\mu = 1, \cdots, n \leqq 6),$$

so that they maximize the integral

* Hotelling assumes that V is to be maximized, but assumes $u(t)$ and $p(t)$ to be known functions; see *A general mathematical theory of depreciation*, loc. cit., p. 343. In a paper entitled *A general problem of maximizing an integral with discontinuous integrand* to be offered to the Transactions of this Society, I have given the mathematical analysis necessary to solve the problem under the assumption that V is to be maximized.

$$I = \int_{t_1}^{t_2} [y_1 y_2 - Q(y_1, y_1', y_2, y_2', p_1, \cdots, p_m, t)] e^{-\int_{t_1}^{t} \delta(v)dv} dt,$$

with $p_1, \cdots, p_m$ considered as known functions of t.

In order to obtain a solution we assume further that

(1) the functions $y_i(t)$, $(i=1, 2)$, defining the maximizing arc E_{12} are continuous on the interval $t_1 t_2$, and this interval can be sub-divided into a finite number of parts on each of which the functions have continuous derivatives,

(2) in a neighborhood R of the values t, y_1, y_1', y_2, y_2' on the maximizing arc E_{12} the functions Q and ϕ have continuous derivatives up to and including those of the second order, and

(3) the functions θ_μ have continuous derivatives up to and including those of the second order near the end-values $(t_1, y_1(t_1), y_2(t_1), t_2, y_1(t_2), y_2(t_2))$, and at these values the $6n$-dimensional matrix

$$\| \; \theta_{\mu t_1} \quad \theta_{\mu y_i(t_1)} \quad \theta_{\mu t_2} \quad \theta_{\mu y_i(t_2)} \; \| \text{ has rank } n.$$

Under these hypotheses, the analysis for the problem of Lagrange with variable end-values as arguments of the integrand applies to the problem of depreciation. I have shown in another paper* that the solution must be such that

$$\text{(a)} \quad F_{y_i'} = \int_{t_1}^{t} F_{y_i} dt + C_i, \qquad (i = 1,2), \text{ along the arc } E_{12},$$

where

$$F(y_1, \cdots, y_2', t, t_1) = \lambda_0 f(y_1, \cdots, y_2', t, t_1) + \lambda_1(t)\phi(y_1, \cdots, y_2', t),$$

where $f(y_1, y_1', y_2, y_2', t, t_1)$ is the integrand of I, where the C_i are arbitrary, and where the λ are Lagrange multipliers;

(b) At the end-points all determinants of order $n+1$ of the matrix

* Roos, *A general problem of minimizing an integral with discontinuous integrand*, to be offered to the Transactions of this Society.

$$\left\| \begin{matrix} H(t_1)-y_{k'}(t_1)H_{y_k'}(t_1) & H_{y_i'}(t_1) & H(t_2)-y_{k'}(t_2)H_{y_k'}(t_2) & H_{y_i'}(t_2) \\ \theta_{\mu t_1} & \theta_{\mu y_i(t_1)} & \theta_{\mu t_2} & \theta_{\mu y_i(t_2)} \end{matrix} \right\|$$

vanish, where as notation

$$H(t_\sigma) = (\sigma - 2)F(t_1) + (\sigma - 1)F(t_2) + \int_{t_1}^{t_2}[F_{t_\sigma} + F_{y_k(t_\sigma)}y_{k'}(t_\sigma)]dx \qquad (\sigma = 1,2),$$

$$H_{y_i'}(t_\sigma) = (\sigma - 2)F_{y_i'}(t_1) + (\sigma - 1)F_{y_i'}(t_2) + \int_{t_1}^{t_2} F_{y_i(t_\sigma)}dt,$$

k is used as an umbral index with range 1, 2, and subscripts y_i, y_i', and t_σ denote differentiation.

(c) In addition to the preceding conditions (a) and (b) certain second order conditions of the calculus of variations must be satisfied by an arc E_{12} which does actually maximize I. It is true, however, that if an arc E_{12}, which maximizes I does actually exist, it is the one defined by the differential equations $\phi(y_1, y_1', y_2, y_2', t)=0$ and the Euler-Lagrange equations in the Du Bois-Reymond form (a) and having end-points satisfying the end-equations $\theta_\mu(t_1, y_1(t_1), y_2(t_1), t_2, y_1(t_2), y_2(t_2))=0$, $(\mu=1, \cdots, n)$, and the transversality conditions (b). Rather than consider the additional conditions of the calculus of variations, let us assume that there does actually exist a maximizing arc E_{12} for the particular problem under consideration and that t, $u(t)$, $p(t)$ in the space t, u, p give the desired solution.

We desire next to obtain an expression for depreciation, which has been defined as the rate of decrease of value.* To do this let us suppose that a machine is operated at a rate $u(t)$ which maximizes I, so that in the expression (4) the functions $u(t)$ and $p(t)$ and the end-values t_1 and t_2 are those defining the maximizing arc. Then the expression (4) will define the maximum value of the machine at the time t_1, and the value of the machine at any other time t $(t_1 \leqq t \leqq t_2)$ can be obtained by replacing t_1 in (4) by t. We obtain

* H. Hotelling, *A general mathematical theory of depreciation*, loc. cit.

$$(7)\qquad V(t) = \int_t^{t_2} [pu - Q(u,u',p,p',\tau)]e^{-\int_t^\tau \delta(\nu)d\nu}\, d\tau + Ke^{-\int_t^{t_2}\delta(\nu)d\nu}.$$

If we let $\Delta(t) = -dV(t)/dt$ be the depreciation of the machine to its operator, we can write by differentiating (7) with respect to t:

$$(8)\qquad \Delta(t) = -\,dI(t)/dt - \delta(t)Ke^{-\int_t^{t_2}\delta(\nu)d\nu}.$$

5. *Relation to Existing Depreciation Theories.* It is most interesting and instructive to see what further hypotheses must be made to obtain the various depreciation theories now commonly used. If, in equation (2), we suppose the salvage value S to be a point function of the time $t_2 = n$ at which the machine is to be salvaged instead of a functional of the rate of production and price as done in this paper, then on changing the parameter of integration from t to τ equation (2) becomes for $t_2 = n$ and $t_1 = t$

$$(9)\qquad V(t) = \int_t^{n} [pu - \psi(u,u',p,p',\tau)]e^{-\int_t^\tau (\delta\nu)d\nu}\, d\tau + S(n)e^{-\int_t^{n}\delta(\nu)d\nu}.$$

If the cost of production function $\psi(u, u', p, p', \tau)$ is a point function $\psi(\tau)$ instead of the more general function ψ of (9), this formula reduces to that given by Hotelling.*

Now, by (9), $V(n) = S(n)$, and hence the total depreciation of the machine for the period $\tau = t$ to $\tau = n$ is evidently $V(t) - S(n)$. If we desire only simplicity, we may assume that the depreciation per unit time is equal to the average depreciation $[V(t) - S(n)]/n$ and obtain the well known straight line formula for depreciation.

Again, if p, u, and ψ of (9) are constant for a certain number of years of the machine's life and then change abruptly at $t_2 = n$ in such a way that it is evident that the end of the useful life of the machine has come, and if the

* See citation (1), p. 343.

force of interest $\delta(t)$ is a constant equal to δ, we can perform the integration in (9) and write

$$(10)\qquad V(t) = [pu - \psi]\frac{1 - e^{-\delta(n-t)}}{\delta} + Se^{-\delta(n-t)}.$$

Now, when $\delta(t) = \delta$, a constant, the discount factor $e^{-\delta}$ is equal to $(1+i)^{-1}$, where i is the rate of interest in the ordinary sense. We obtain, therefore,

$$(11)\quad V(t) = [pu - \psi]\frac{1 - (1 + i)^{-(n-t)}}{\delta} + S(1 + i)^{-(n-t)},$$

and, at $t=0$, this becomes

$$(12)\qquad V(0) = [pu - \psi]\frac{1 - (1 + i)^{-n}}{\delta} + S(1 + i)^{-n}.$$

If we eliminate $[pu-\psi]/\delta$ from the equations (11) and (12), we obtain

$$\begin{vmatrix} V(t) - S(1 + i)^{t-n} & 1 - (1 + i)^{t-n} \\ V(0) - S(1 + i)^{-n} & 1 - (1 + i)^{-n} \end{vmatrix} = 0.$$

By adding $-S$ times the second column to the first column, then subtracting the second row from the first row and expanding, we obtain at once

$$V(0) - V(t) = [V(0) - S(n)]\frac{(1 + i)^t - 1}{(1 + i)^n - 1}\cdot$$

When we introduce the customary notation

$$s_{\overline{t|}} = \frac{(1 + i)^t - 1}{i},$$

this formula becomes

$$(13)\qquad V(0) - V(t) = [V(0) - S(n)]\frac{s_{\overline{t|}}}{s_{\overline{n|}}},$$

which is the well known formula for the accumulation of depreciation allowances at the end of the tth year under the sinking-fund method.*

* Rietz, Crathorne and Rietz, *Mathematics of Finance*, pp. 112–121.

If now the formula for $V(t+1)$ obtained from (13) by replacing t by $t+1$ be subtracted from (13), we obtain

$$V(t) - V(t+1) = [V(0) - S(n)]\frac{(1+i)^t}{s_{\overline{n}|}}. \tag{14}$$

This formula states that the depreciation for any year is equal to the depreciation charge for the first year at compound interest at the rate of i per annum. This is, therefore, the so-called "compound interest method" of providing for depreciation.

If we perform the indicated differentiation in (8), we obtain on solving for p

$$p = \frac{Q + \delta K e^{-\delta(n-t)} - dV(t)/dt}{u}, \tag{15}$$

which is equivalent to the formula by which J. S. Taylor defines *unit cost plus*.*

From the general theory of this paper we have thus derived the popular "sinking fund" or "equal annual payment" formula for depreciation (13) and the "compound interest formula" (14) by assuming that the rate of production u, the price p and the cost of production ψ are constant throughout the useful life of the machine. We have also seen that the unit cost plus formula can be obtained directly from the formula for depreciation as given in this paper when the price p and the rate of production u are known functions of the time.

6. *Some Generalizations.* If instead of the hypothesis of §3, we adopt Hotelling's hypothesis that the operator of a machine will do everything in his power to maximize the present value $V(t_1)$ of his machine, the problem of depreciation for a fixed t_1 is a Mayer problem of a general type considered by Bliss.† When t_1 is variable this prob-

* J. S. Taylor, *A statistical theory of depreciation based on unit cost*, Journal of the American Statistical Association, December, 1923.

† Bliss, *The problem of Mayer with variable end points*, Transactions of this Society, vol. 19 (1918).

lem can be made over into the Bliss problem by introducing a new variable z satisfying the differential equation $z=0$ and the end condition $z(t_1)=t_1$. It is not, however, necessary to complicate matters by introducing another variable to obtain the form given by Bliss, for, if we take the term in K in (4) under the integral sign, we obtain

$$V(t_1) = \int_{t_1}^{t_2}\left[(pu - Q)e^{-\int_t^{\tau}\delta(\nu)\delta\nu} + \frac{Ke^{-\int_{t_1}^{t_2}\delta(\nu)d\nu}}{t_2 - t_1}\right]d\tau,$$

and this is a special case of the general problem referred to in §3.* For the conditions (a) and (b) of §3 the function $f(u, u', p, p', t, t_1)$ of that section becomes a function $f(u, u', p, p', t, t_1, t_2)$ identically equal to the above integrand.

I have already discussed a related problem for the case of several competing machines and I believe that the analysis given there can be remodeled after the methods of this paper to fit the depreciation problem for several competing machines.†

There is another important possible direction of extension of this paper. As I have pointed out in §2, there are instances when the more general functional equation of demand as given by (5) should replace the differential equation of demand (6). The resulting problem is a problem in the maxima of functionals which would be well worth special investigation.‡

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* Roos, *General problem of minimizing an integral with discontinuous integrand*, loc. cit. See also E. H. Clarke, *On the minimum of the sum of a definite integral and a function of a point*, Doctoral Dissertation, University of Chicago, and Miss Sinclair, Annals of Mathematics, (2), vol. 10, pp. 55–80.

† Roos, *Generalized Lagrange problems in the calculus of variations*, Transactions of this Society, not yet published.

‡ For a related problem in the maxima of functionals, see Hahn, *Über die Lagrange'sche Multiplicatorenmethode*, Sitzungsberichte der Akademie Wien, vol. 131 (1922), pp. 531–550.