NOTE ON A CONVERGENCE PROOF

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Some years ago I published a particularly simple proof of the convergence of the Fejér mean of the Fourier series for an arbitrary continuous function.* I did not notice until some time later that the same proof had already been given by Haar† in his thesis. The present note constitutes a renewed attempt to contribute something to the theory of the method in question, by applying it to a problem which is not treated by Haar, in the passage cited at any rate. The substance of the note consists in the proof of the following theorem:‡

Let $f(x)$ be an arbitrary continuous function of period 2π. With each positive integral value of n, let an integer m_n be associated, subject merely to the condition that $m_n \geqq n$, and let

$$\tau_n(x) = \frac{1}{nm_n} \sum_{i=1}^{m_n} f(t_i) \frac{\sin^2 \frac{1}{2}n(t_i - x)}{\sin^2 \frac{1}{2}(t_i - x)}, \tag{1}$$

where $t_i = 2i\pi/m_n$. Then $\tau_n(x)$ converges uniformly toward $f(x)$ as n becomes infinite.

The reasoning is given in full, so that it can be understood

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* *Note on a method of proof in the theory of Fourier's series*, this Bulletin, vol. 27 (1920–21), pp. 108–110.

† A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Dissertation, Göttingen, 1909; p. 29; reprinted in Mathematische Annalen, vol. 69 (1910), pp. 331–371; pp. 353–354.

‡ For the case $m_n = n$, see D. Jackson, *A formula of trigonometric interpolation*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 371–375; S. Bernstein, *Sur la convergence absolue des séries trigonométriques*, Comptes Rendus, vol. 158 (1914), pp. 1661–1663; L. Fejér, *Über Interpolation*, Göttinger Nachrichten (1916), pp. 66–91; pp. 87–91. For a corresponding generalization of the ordinary formula of trigonometric interpolation, see D. Jackson, *Some notes on trigonometric interpolation*, American Mathematical Monthly, vol. 34 (1927), pp. 401–405. For the underlying idea of the present treatment, see also Hahn, *Über das Interpolationsproblem*, Mathematische Zeitschrift, vol. 1 (1918), pp. 115–142.

without reference to Haar's paper. By a well known identity,

$$\frac{\sin^2 \frac{1}{2}n(t_i - x)}{\sin^2 \frac{1}{2}(t_i - x)} = n + 2(n-1)\cos(t_i - x)$$

$$(2)\quad + 2(n-2)\cos 2(t_i - x) + \cdots + 2\cos(n-1)(t_i - x)$$

$$= n + 2\sum_{k=1}^{n-1}(n-k)(\cos kt_i \cos kx + \sin kt_i \sin kx),$$

so that $\tau_n(x)$ is a trigonometric sum of order $n-1$ in x. Since $n-1<m_n$, and since

$$\sum_{i=1}^{m_n} \cos kt_i = \sum_{i=1}^{m_n} \sin kt_i = 0$$

for $0<k<m_n$, it is seen that

$$(3)\qquad \frac{1}{nm_n}\sum_{i=1}^{m_n}\frac{\sin^2 \frac{1}{2} n(t_i - x)}{\sin^2 \frac{1}{2}(t_i - x)} = 1.$$

This may also be expressed by saying that if $f(x)\equiv 1$, the corresponding $\tau_n(x)$ is identically equal to 1 for all values of n.

To take another very special case, let $f(x)$ be of the form $\cos px$, where p is a given positive integer, and let the form of the corresponding $\tau_n(x)$ be determined with the aid of (2). The expression $\sum_i \cos pt_i \sin kt_i$ is equal to zero for all values of k. The question ultimately at issue being one of convergence for $n=\infty$, it is sufficient to consider values of $n>2p$. Then $p<m_n/2$, and $\sum_i \cos^2 pt_i = m_n/2$. Under the hypotheses, $p+n-1$ may or may not be less than m_n. If $p+n-1<m_n$, $\sum_i \cos pt_i \cos kt_i = 0$ for all the values of k (including $k=0$) that come into consideration, except $k=p$, and $\tau_n(x)$ reduces to a single term:

$$(4)\qquad \tau_n(x) = \frac{n-p}{n}\cos px.$$

If $p+n-1\geqq m_n$, there is one other term, resulting from the fact that $\sum_i \cos pt_i \cos(m_n-p)t_i = m_n/2$, and

$$(5)\quad \tau_n(x) = \frac{n-p}{n}\cos px + \frac{n-m_n+p}{n}\cos(m_n-p)x.$$

But $n-m_n\leqq 0$, and $(n-m_n+p)/n\leqq p/n$, which approaches zero as n becomes infinite. So, whichever of the expressions (4), (5) may be in force from time to time as n takes on successive values, it is clear that

$$\lim_{n=\infty} \tau_n(x) = \cos px,$$

uniformly for all values of x. There is a corresponding proof if $f(x)\equiv\sin px$.

On the other hand, the $\tau_n(x)$ corresponding to the sum of any finite number of functions is the sum of the τ's constructed for the various functions separately, and converges if each of the latter τ's is convergent. *So $\tau_n(x)$ converges uniformly toward $f(x)$, whenever $f(x)$ itself is identically a trigonometric sum.*

In transition, it is to be noted from (1) and (3) that $|\tau_n(x)|\leqq M$, if M is the maximum of $|f(x)|$.

Finally, let $f(x)$ be an arbitrary continuous function of period 2π. Let ϵ be an arbitrary positive quantity. By Weierstrass's theorem there exists a trigonometric sum $T(x)$ such that

$$|f(x) - T(x)| \leqq \epsilon/3 \tag{6}$$

for all values of x. If $\tau_n(x)$ is defined by (1), and if $\tau_{n1}(x)$ is similarly formed with $T(t_i)$ in place of $f(t_i)$, it follows from the preceding paragraph, applied to the difference $T(x)-f(x)$, that

$$|\tau_{n1}(x) - \tau_n(x)| \leqq \epsilon/3 \tag{7}$$

for all values of n and x. And by the italics at the end of the second paragraph preceding,

$$|T(x) - \tau_{n1}(x)| \leqq \epsilon/3 \tag{8}$$

if n is sufficiently large. For such values of n, by combination of (6), (7), and (8), $|f(x) - \tau_n(x)| \leqq \epsilon$, which is equivalent to the conclusion of the theorem.

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