EXPANSION IN SERIES OF NON-INVERTED FACTORIALS*

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An expansion of the form

$$f(z) = \sum \frac{b_n}{(z+1)(z+2)\cdots(z+n)}$$

can be obtained from the consideration of Cauchy's formula

$$2\pi i f(z) = \int_C \frac{f(t)dt}{z-t},$$

if $f(z)=0$ at infinity, together with the result†

$$(1) \qquad \frac{n!}{(z+1)(z+2)\cdots(z+n+1)} = \int_0^1 u^n(1-u)^z du,$$

where $(1-u)^z$ denotes the branch reducing to unity for $u=0$. The above relations can also be used for deriving an expansion in series of non-inverted factorials. By (1) we have

$$\frac{1}{z-t} = \int_0^1 (1-u)^{z-t-1}du.$$

Consider

$$(1-u)^{z-t-1} = (1-u)^z(1-u)^{-t-1}.$$

Since

$$(1-u)^z = 1$$

when $u=0$, we may write

$$(2) \qquad (1-u)^z = 1 - \frac{z}{1!}u + \frac{z(z-1)}{2!}u^2 - \cdots + \frac{(-1)^n}{n!}z(z-1)\cdots(z-n+1)u^n + \cdots.$$

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* Presented to the Society, September 9, 1927.

† Whittaker and Watson, *Modern Analysis*, 3d edition, Cambridge University Press, 1920, p. 144.

The binomial expansion (2) will be uniformly convergent for $0 \leqq u \leqq 1$ when $R(z) > 0$.* Also let us introduce the condition

$$R(-t-1) > 0,$$

that is,

$$R(t) < -1.$$

Then the expansion

$$(3) \qquad (1-u)^{z-t-1} = (1-u)^{-t-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} u^n (1-u)^{-t-1} z(z-1) \cdots (z-n+1),$$

which holds for $0 \leqq u \leqq 1$, can be integrated termwise, so that we may write

$$(4) \qquad \frac{1}{z-t} = \int_0^1 (1-u)^{z-t-1} du = \int_0^1 (1-u)^{-t-1} du + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cdot \left[\int_0^1 u^n (1-u)^{-t-1} du \right] \cdot z(z-1) \cdots (z-n+1).$$

But

$$\int_0^1 u^n (1-u)^{-t-1} du = \frac{-n!}{t(1-t)(2-t) \cdots (n-t)},$$

and hence we have

$$(5) \qquad \frac{1}{z-t} = -\frac{1}{t} - \sum_{n=1}^{\infty} \frac{(-1)^n z(z-1)(z-2) \cdots (z-n+1)}{t(1-t)(2-t) \cdots (n-t)},$$

where $R(z) > 0$, $R(t) < -1$.

Let $R_n(u)$ denote the remainder after $(n+1)$ terms of the series (2) multiplied by $(1-u)^{-t-1}$. Since (2) is uniformly convergent, given ϵ, n_0 can be found so that for $n \geqq n_0$ and all u, $0 \leqq u \leqq 1$, we have

$$| R_n(u) | < \epsilon.$$

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* $R(z)$ denotes the real part of z.

If we let $R_n'(t)$ denote the remainder after $(n+1)$ terms of the series (5), we may observe that

$$R_n'(t) = \int_0^1 R_n(u)du,$$

and hence

$$|R_n'(t)| \leqq \int_0^1 |R_n(u)|\, du < \epsilon,$$

where ϵ is independent of t; consequently, (5) is a uniformly convergent series in t.

Let $f(z)$ be a function analytic on and outside a closed contour C situated to the left of $R(z) = -1$, and vanishing at infinity; then

$$\begin{aligned} -2\pi i f(z) &= -\int_C \frac{f(t)dt}{z-t} \\ &= \int_C f(t)\left[\frac{1}{t} + \sum_{n=1}^{\infty}(-1)^n \frac{z(z-1)\cdots(z-n+1)}{t(1-t)\cdots(n-t)}\right]dt \end{aligned}$$

when $R(z) > 0$. Since integration termwise is justifiable, we have

$$(6) \qquad \begin{aligned} -2\pi i f(z) = \int_C \frac{f(t)dt}{t} + \sum_{n=1}^{\infty}(-1)^n \\ \cdot\left(\int_C \frac{f(t)dt}{t(1-t)\cdots(n-t)}\right) z(z-1)\cdots(z-n+1). \end{aligned}$$

Hence we may state the following theorem.

THEOREM I. *Let $f(z)$ be a function analytic on and outside of a closed contour C situated to the left of $R(z) = -1$, and vanishing at infinity; then for all z with $R(z) > 0$*

$$(7) \qquad \begin{aligned} f(z) = b_0 + b_1 z + b_2 z(z-1) + \cdots \\ + b_n z(z-1)\cdots(z-n+1) + \cdots, \end{aligned}$$

where

$$(8) \qquad b_n = \frac{(-1)^{n+1}}{2\pi i}\int_C \frac{f(t)dt}{t(1-t)\cdots(n-t)}.$$

If we take $(1-u)^{z-t-1}$ as

$$(1-u)^{z+k}\cdot(1-u)^{-t-1-k},$$

where k may be complex, repeat the steps by means of which Theorem I was deduced, and replace z by $z+k$ and t by $t+k$, we find the following generalized theorem.

THEOREM II. *Let $f(z)$ be a function analytic on and outside of a closed contour C situated to the left of $R(z)=-R(1+k)$, and vanishing at infinity, then for all z with $R(z)>-R(k)$, we have*

$$(9)\quad f(z) = b_0 + b_1(z+k) + b_2(z+k)(z+k-1)+\cdots + b_n(z+k)(z+k-1)\cdots(z+k-n+1)+\cdots,$$

where

$$(10)\quad b_n=\frac{(-1)^{n+1}}{2\pi i}\int_C \frac{f(t)dt}{(t+k)(1-t-k)(2-t-k)\cdots(n-t-k)}.$$

Let U be max. $|f(t)|$ on C and l the length of C; then considering the expansion defined by Theorem I, it is observed that $t=t_1+it_2$ has $-t_1>1$, since $R(t)<-1$ so that $|n-t|\geqq n-t_1>n+1$, and hence

$$\frac{1}{|t(1-t)\cdots(n-t)|}<\frac{1}{(n+1)!}.$$

Consequently

$$(11)\qquad |b_n|<\frac{h}{(n+1)!}. \qquad \left(h=\frac{Ul}{2\pi}\right).$$

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