$$e = 1 - \frac{1}{2\pi i}\int_C \left(\frac{d\theta}{dz}\right)\cdot\left[\frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]dz$$

$$= 1 - \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{N''(\theta) + iN'(\theta)}{N'(\theta) + iN(\theta)} - \frac{N'(\theta)}{N(\theta)}\right]d\theta,$$

since θ varies from 0 to 2π when z describes C. Moreover

$$e = 1 - \frac{1}{2\pi i}\cdot\left\{\log\left[N'(\theta) + iN(\theta)\right]\right\}_0^{2\pi} + \frac{1}{2\pi i}\left[\log N(\theta)\right]_0^{2\pi}.$$

We know that $N(\theta)$ is real; hence $[\log N(\theta)]_0^{2\pi}=0$. On the other hand, the variation of the argument of $[N'(\theta)+iN(\theta)]$ as θ changes from 0 to 2π is zero, so that $\log\,[(N'(\theta)+iN(\theta))]_0^{2\pi}=0$. Hence $e=1$. This proves the theorem.

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LINEAR INEQUALITIES IN GENERAL ANALYSIS*

BY L. L. DINES

1. *Introduction.* In his studies in general analysis, E. H. Moore has developed† a theory of the linear functional equation

$$\xi + J\kappa\xi = \eta.$$

Here ξ and η denote functions (the latter given, the former to be determined) belonging to a class $\mathfrak{M}$ of real-valued functions on a general range $\mathfrak{P}$. The kernel function κ belongs to a class $\mathfrak{K}$ which is well defined in terms of the fundamental class $\mathfrak{M}$. A sufficient foundation for the theory is laid by means of postulates upon the class $\mathfrak{M}$ and the functional operation J.

The purpose of the present paper is to consider the linear *inequality*

$$(1) \qquad \xi + J\kappa\xi > 0,$$

* Presented to the Society, September 8, 1927.

† *On the foundations of the theory of linear integral equations*, this Bulletin, vol. 18, pp. 334–362.

a type which includes the *non*-homogeneous inequality in a sense indicated in § 5.

The most interesting feature is the relationship between the inequality (1) and the *adjoint equation*

$$\mu + J\mu\kappa = 0,$$

as expressed in the theorem of § 4. For the proof of this theorem, two postulates are introduced in addition to those used by Professor Moore.

2. *The Basis*. As a basis we assume Moore's $\sum_5$:*

$$(\mathfrak{A}\ ;\ \mathfrak{P}\ ;\ \mathfrak{M}\ ;\ \mathfrak{K} \equiv (\mathfrak{M}\mathfrak{M})_*\ ;\ J \text{ on } \mathfrak{K} \text{ to } \mathfrak{A}),$$

where $\mathfrak{A}$ is the class of real numbers; $\mathfrak{P}$ is a general class of elements p; $\mathfrak{M}$ is a class of functions (μ, ξ, η, π) on $\mathfrak{P}$ to $\mathfrak{A}$, having the properties L, C, D; and the operator J has the properties L and M.

3. *An Equation Equivalent to the Inequality*. The inequality (1) is obviously equivalent to an equation

$$(2) \qquad \xi + J\kappa\xi = \pi,$$

where π is any function of $\mathfrak{M}$ which is everywhere positive on $\mathfrak{P}$. This situation suggests an additional postulate upon the class $\mathfrak{M}$, namely that it contain at least one positive function. We shall find it desirable presently to make a more comprehensive postulate.

Equation (2) is of the Fredholm type treated by Moore. Its solution depends upon the Fredholm determinant F_κ of the kernel κ. In case $F_\kappa \neq 0$, the equation admits the solution

$$(3) \qquad \xi = \pi + J\lambda\pi,$$

* Loc. cit., p. 349. We might have assumed the basis $\sum_6$ (p. 352) which differs from $\sum_5$ in the presence of two classes of functions $\mathfrak{M}'$ and $\mathfrak{M}''$ on two conceptually distinct ranges $\mathfrak{P}'$ and $\mathfrak{P}''$. The basis $\sum_6$ reduces to $\sum_5$ if $\mathfrak{P}' = \mathfrak{P}''$ and $\mathfrak{M}' = \mathfrak{M}''$. Our choice is in the interest of typographical simplicity. We were influenced also by the fact that we have discovered no instance involving distinct ranges $\mathfrak{P}'$ and $\mathfrak{P}''$ in which our postulate E (§4) is satisfied.

where λ is the reciprocal kernel (resolvent) for κ. The inequality (1) then admits, in this case, the solution (3) where π is any positive function of $\mathfrak{M}$.

Of more interest is the exceptional case in which $F_\kappa=0$. The theory of equation (2) for this case (without the restriction that π be positive) has been treated in detail by Hildebrandt.* The facts are entirely analogous to those well known in the theory of ordinary integral equations. The equation (2) in general admits no solution. The corresponding homogeneous equation

$$\xi + J\kappa\xi = 0$$

admits a finite number of linearly independent solutions $\xi_1, \xi_2, \cdots, \xi_m$; and the adjoint homogeneous equation

$$\mu + J\mu\kappa = 0 \tag{4}$$

admits the same number of linearly independent solutions $\mu_1, \mu_2, \cdots, \mu_m$. A necessary and sufficient condition that the equation (2) admit a solution is that π be orthogonal to each of the functions μ_i. That is, that

$$J\mu_i\pi = 0, \qquad (i = 1,2, \cdots ,m). \tag{5}$$

The solution of (2) is then

$$\xi = \pi + J\lambda\pi + \sum_{i=1}^{m} c_i\xi_i$$

where λ is a pseudo-resolvent for κ, and the c_i are arbitrary constants.

To apply these results to the inequality (1), we note that the kernel κ is given; and if $F_\kappa=0$, the set of fundamental solutions $\mu_1, \mu_2, \cdots, \mu_m$ of the adjoint equation (4) can be determined. The problem of the inequality then reduces to the determination of a positive function π in $\mathfrak{M}$ which satisfies conditions (5).

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* *On pseudo-resolvents of linear integral equations in general analysis*, Annals of Mathematics, vol. 21 (1920), pp. 323–330.

The machinery which suffices for the theory of the functional equation apparently affords no means of attack on this problem. We therefore introduce in the next section new postulates which are sufficient to secure our desired result.

4. *Two New Postulates.* The postulates to be introduced involve a certain property relative to functions, which has been used in an earlier paper,* and which may be defined as follows:

A real-valued function is said to be *M-definite* if it is somewhere positive and nowhere negative, or somewhere negative and nowhere positive on its range; in other words if it is not identically zero and does not change sign.

Postulate N: If μ is an M-definite function of $\mathfrak{M}$ and π is a positive function of $\mathfrak{M}$, then $J\mu\pi \neq 0$.

Postulate E: If $(\mu_1, \mu_2, \cdots, \mu_m)$ is a set of functions of $\mathfrak{M}$ such that no linear combination of them is M-definite, then there exists a positive function π in $\mathfrak{M}$ such that

$$J\mu_i\pi = 0, \qquad (i = 1, 2, \cdots, m). \tag{5}$$

The postulate N is a natural generalization of a simple property possessed by the operator J in each of Moore's classical instances (I), (II), (III), and (IV).

The existential postulate E is not perhaps so natural, and it undoubtedly invites further analysis. It is however satisfied in each of the four classical instances, as has been shown elsewhere by the author.† This postulate (together

* See *On sets of functions of a general variable*, Transactions of this Society, vol. 29 (1927), pp. 463–470.

† For the instance (II), including (I), see *Note on certain associated systems of linear equalities and inequalities*, Annals of Mathematics, vol. 28 (1926), p. 41, Theorem II.

For the instance (III), the proof is contained in a paper entitled *A theorem on orthogonal sequences* which will probably be published at an early date.

For the instance (IV), see *A theorem on orthogonal functions with an application to integral inequalities*, which is to appear soon in the Transactions of this Society.

An equivalent (and in some respects preferable) formulation of the postulate may be given in terms of the notion M-rank defined in the paper with the assumption that $\mathfrak{M}$ has the property L) implies the existence of at least one positive function in $\mathfrak{M}$.

5. *The Inequality and its Adjoint Equation.* By use of the postulates just introduced, we now obtain the following

THEOREM: *The inequality*

$$(1) \qquad \xi + J\kappa\xi > 0$$

admits a solution, if and only if the adjoint equation

$$(6) \qquad \mu + J\mu\kappa = 0$$

admits no M-definite solution.

First, we note that the case $F_\kappa \neq 0$ is consistent with the theorem, since in that case the inequality (1) admits a solution (3), and the equation (6) admits only the solution $\mu = 0$ which is not M-definite.

Suppose then that $F_\kappa = 0$, and consider a fundamental set of solutions

$$(7) \qquad \mu_1, \mu_2, \cdots, \mu_m$$

of the adjoint equation (6). That is, the linear combinations of this set comprise the totality of solutions of (6). There are then two possibilities.

If the set (7) admits no linear combination which is M-definite, then the equation (6) admits no M-definite solution, while the inequality (1) admits a solution, since by postulate E there is a positive function π satisfying (5).

If, on the other hand, the set (7) admits an M-definite linear combination, say $\mu^* = \sum c_i\mu_i$, then the equation (6) admits the M-definite solution $\mu = \mu^*$. And the inequality (1) admits no solution. For suppose it did admit a solution $\xi = \xi^*$, that is suppose $\xi^* + J\kappa\xi^* = \pi$, where π is a positive function. Then π must satisfy the conditions $J\mu_i\pi = 0$,

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referred to in the preceding footnote, namely: If $(\mu_1, \mu_2, \cdots, \mu_m)$ is a set of functions of M-rank zero, then there exists a positive function π in $\mathfrak{M}$ such that $J\mu_i\pi = 0$, $(i = 1, 2, \cdots, m)$.

and since J has the property L (linearity) it follows that $J\mu^*\pi=0$, which contradicts postulate N.

We have shown that (1) admits a solution in those and only those cases in which (6) admits no M-definite solution. This establishes the theorem.

6. *The Non-Homogeneous Inequality.* The inequality

$$\xi + J\kappa\xi > \eta \tag{8}$$

where η is a given function of $\mathfrak{M}$, may be replaced by the two simultaneous inequalities

$$\begin{aligned} \xi + J\kappa\xi - \eta x &> 0, \\ x &> 0, \end{aligned} \tag{9}$$

where x is a real number to be determined. For any solution $\xi=\xi^*$ of (8) determines a solution ($\xi=\xi^*$, $x=1$) of (9), and conversely any solution ($\xi=\xi^*$, $x=x^*$) of (9) determines a solution $\xi=\xi^*/x^*$ of (8).

By the method of adjunctional composition (see Moore, loc. cit., p. 355), the system (9) may be written in the homogeneous form

$$\xi' + J\kappa'\xi' > 0$$

relative to a new basis $\sum_5'$:

$$(\mathfrak{A}\ ;\ \mathfrak{P}'\ ;\ \mathfrak{M}'\ ;\ \mathfrak{K}' \equiv (\mathfrak{M}'\mathfrak{M}')^*\ ;\ J' \text{ on } \mathfrak{K}' \text{ to } \mathfrak{A}).$$

This new basis is the adjunctional composite of our general basis $\sum_5$, and the particular and very simple basis

$$(\mathfrak{A}\ ;\ \mathfrak{P}^{\mathrm{I}}\ ;\ \mathfrak{M}^{\mathrm{I}} \equiv \mathfrak{A}\ ;\ \mathfrak{K}^{\mathrm{I}} \equiv \mathfrak{A}\mathfrak{A}\ ;\ J^{\mathrm{I}} \text{ on } \mathfrak{K}^{\mathrm{I}} \text{ to } \mathfrak{A}),$$

in which $\mathfrak{P}^{\mathrm{I}}$ is a class containing a single element, $\mathfrak{M}^{\mathrm{I}}$ is the class of real numbers, and J^{I} is the identity operation.

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