CULLIS ON MATRICES

Matrices and Determinoids. By C. E. Cullis. University of Calcutta Readership Lectures. Cambridge University Press, Vol. III, Part 1, 1925. xviii+681 pp.

The first two volumes of this treatise were reviewed in this Bulletin (vol. 26 (1920), pp. 224–233). The present volume continues the development, which has become more extensive than at first contemplated, so that the third volume has had to appear in two parts. The second part, it is stated, will deal chiefly with structural matrices. The author's avowed aim is to carry forward the program of Sylvester, who conceived a magnificent treatise on universal algebra, which he never carried out. The present treatise to a large extent is realizing this program. To Professor Cullis, matrices means practically all of algebra, and it is easy to see from his standpoint that every expression in algebra, whether it depends upon one or upon more variables, is in reality a function of some sort of matrix. To the reviewer this is tantamount to saying that every function of one or more variables is ipso facto a function of a vector of a space of the same number of dimensions, or of various vector products in such space. The difference is simply one of point of view or mode of statement. It is simpler, in the reviewer's opinion, to make the whole subject a branch of the theory of hypernumbers, than to build up a theory of "sets," although this is really following the tradition of Sir W. R. Hamilton, the creator of quaternions. The author insists rightly upon the matrix as a single entity, and considers that a *pure* calculus will be determined by a system of scalar numbers, so that he would introduce hypernumbers later, the ordinary quaternion, for instance, being a product of a scalar matrix and a matrix whose elements are the usual 1, i, j, k. While this is logical, and is a justifiable line of development, it seems to the reviewer to complicate the subject unnecessarily. The author further considers that on the applied side, every physical entity can be represented by a matrix, and he certainly would find plenty of justification for this view in modern physics. His assertion (p. vi) that the expressions would be two-dimensional tables, hardly seems tenable, however. For witness the modern tensor theory.

The scope of this volume is best indicated by a brief indication of the contents of the various chapters. The first three (XX, XXI, XXII) deal with rational integral functions of any number of variables, irresoluble and irreducible functions and factors, common factors, resultants, eliminants, discriminants, and common roots. Symmetric functions are considered specially in Chapter XXII. In Chapter XXIII begins the consideration of rational integral functional matrices, that is, matrices whose elements are rational integral functions of scalar variables, $x, y, z, \cdots$. The definition is given and properties proved of irreducible and irresoluble divisors of such matrices. These divisors are rational integral functions of the same variables as the matrix, and are of order i when they will divide every minor determinant of order i which arises from the matrix. A *regular* minor determinant of order i is one which, with respect to an irreducible or irresoluble divisor t, does not vanish identically, and is not divisible by a higher power of t than that power which is the highest that will divide all minors of the order i. The terms invariant factors, and elementary divisors, of the classic literature, are replaced in this chapter by *potent divisor*. The greatest common divisors D are called maximum factors, the invariant factors E are called maximum divisors, the divisors of these are called potent factors and potent divisors. These are all rational in the domain considered. The former elementary divisors might have been irrational. The potent factors and divisors of the product of two matrices are considered. Following this the same divisors are considered for *compartite* matrices, that is, matrices which may be considered to be the *direct sum* (in the hypernumber sense) of simpler matrices. The next chapter, XXIV, takes up the treatment of *equipotent* matrices, that is, matrices which are *equivalent* in the sense of having the same rank, and the same potent divisors. The equipotent transformations are discussed in detail. With regard to these topics the reviewer will remark only that the fundamental elements of a matrix are what he has previously called its *shear regions*. When these are given everything is virtually given and it would be simpler to treat the subject on such a basis. The two chapters just discussed introduce the idea of rationality for a given domain, which is after all an *arithmetic* notion as contrasted with an algebraic notion. The reviewer has in mind that algebra must deal with what he calls *hypernumbers*, which are frequently called in France *relative numbers*, that is to say, numbers with qualitative signs. Even the introduction of $+$ and $-$ was necessary to produce *algebra*. Mere literal notations do not take one out of the region of *arithmetic*. In Arithmetic we find the root of the notion of *domain of rationality*. Its extension to algebraic fields followed. Thus in all discussions of such topics as we have before us we find these two aspects, the hypernumber structure, and the domain of rationality, or sometimes of integrity, as for instance in Dickson's Arithmetic of Algebras.

Chapter XXV is concerned with rational integral functions of a square matrix. It introduces the notions of *latent roots*, and the *characteristic matrix*, which is the matrix $\phi-\lambda I$, where I is the identity matrix, and the elements of the matrix ϕ are constant. It includes also the determination of all *rational integral equations* satisfied by ϕ. At the end of the chapter is Frobenius' solution of the matrix equation $\psi^2=\phi$. The Hamilton-Cayley equation is developed.

Chapter XXVI develops the subject of *equimutant transformations*, or transformations of the form $A(\ )A^{-1}$ where A is undegenerate. This transformation corresponds directly to what is usually called a transformation in the theory of groups, or in the theory of linear homogeneous groups to an orthogonal transformation. It might be added it is also in the general theory of vectors in N dimensions, a rotation. It enables us to bring a matrix with constant elements to the canonical form. Also the idea of *transmute* is introduced. By this is meant that if $\phi=\alpha\beta$ where α and β are of rank r, the first with m rows and r columns, the second with

r rows and m columns, then $\phi_1 = \beta\alpha$ is the transmute of ϕ, a notion connecting with previous results (vol. I, page 154) and useful in stating the theorem that two square matrices with constant coefficients are connected by an equimutant transformation when and only when they have a common first transmute. Transmutes are not unique, there being an infinity of first transmutes for the matrix ϕ, which, however, are all transformable into one another by equimutant transformations. In fact, the best way to arrive at a first transmute is to consider the matrix as a linear vector operator which annuls a region of dimensions equal to the nullity, and if for the region left κ is the identity matrix, then a first transmute of ϕ is $\kappa\phi\kappa$. Or in the array form, if we transform the matrix so that it has zero columns equal in number to the nullity, then by erasing the same number of rows at the bottom, producing a square matrix, we will have a first transmute. There is much material in this chapter which is related to the characteristic equation of a matrix, including the standard theorems.

Chapter XXVII deals with *commutants*. A commutant X is a matrix solution of the equation $AX = XB$ where A and B are square matrices. It is either a zero matrix or has a determinate structure. If A and B have only constant elements X is non-zero only if A and B have a latent root in common. When A and B are canonical square matrices with the same latent root, X is a general "ruled compound slope." The ideas advanced in connection with this notion have a direct connection with what the reviewer has previously called "*associative units*" (Transactions of this Society, vol. 4 (1903), 251–287). These play an important part in the linear associative algebras. There is a great deal of new material in the chapter which can scarcely be detailed here. It evidently is in preparation for the developments of the second part of volume three. Chapter XXVIII deals with commutants of commutants, which are commutants of every commutant of a given matrix A. They are proved to be the rational integral functions of A. Chapter XXIX deals with invariant transformands which are solutions of the equation $AXB = X$. When there are solutions which are non-zero they have a definite structure. It is evident that if B is invertible the invariant transformands go back into the commutant list.

The theory of matrices, in whatever notation it may be expressed, has a growing importance. From the study of finite matrices have followed many recent developments that involve the properties of infinite matrices. The study of quadratic forms on an infinity of variables, either denumerable or non-denumerable, the various developments of integral equations and of linear operators in general, and other related subjects, all lead back to the fundamental structure of matrices. These in turn are special cases of hypernumbers, and consequently we find that in hypernumbers we have as universal an algebra as we may at present study. The matrices are practically the associative hypernumbers, and special structures in these forms give the special algebras. The non-associative systems however are very much more numerous and very much more complicated. There has been up to the present time little use for such systems, perhaps due to the fact that their algebra has not been worked out, but this will be done in the future. It is evident that from the study of such systems associativity, important as it is, is of trifling importance in the general problems of structure. It may be then that the day will come when the matrix will be of only very limited importance in the study of structural physics, and the non-associative hypernumbers will give us the keys to the universe.

It is nevertheless a matter for congratulation to Professor Cullis that he is carrying out such a heavy piece of work as the present development of matrices, and it is to be hoped that he may bring it to a successful conclusion.

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STENSTRÖM ON THE 27 LINES

Synthetische Untersuchungen des Systems von 27 Geraden einer Fläche dritter Ordnung. By Olof Stenström. Upsala, Appelbergs Boktryckeri Aktiebolag, 1925. 128 pp.

It was a matter of course, in the history of geometry, that the cubic surface should be much studied. The zest with which many have pursued this study was not a matter of course but was largely due to the elegant configuration of the 27 lines on the surface. Since their fortunate discovery by Cayley and Salmon in 1849, they have been essential to nearly all investigations concerning it—investigations in which conspicuous names are Schläfli, Clebsch, Sylvester, Cremona, Sturm, Klein, Reye, Zeuthen, Schur, Henderson, and Baker. No decade has, indeed, been without notable work on the surface of third order. Since 1920 the contributions of two men deserve especial mention. The first was E. Stenfors,* who worked on the projective transformations of a Schläfli double-six into itself, as well as on like transformations of the whole system of lines, and their groups. Olof Stenström followed in 1925 with the booklet which is the subject of this review.

If, contrary to custom, a formal Table of Contents is included in a review, it may perhaps be justified by the fact that the book itself contains none, although such a table would aid both those who wish to read it and those who wish to learn more quickly of its content.

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* *Die Schläflische Konfiguration von zwölf Geraden einer Fläche dritter Ordnung*, 1921. *Über die Geradenkonfiguration einer Fläche dritter Ordnung, bzw. Klasse*, 1922. Both memoirs are contained in Series A, vol. 18, of that valuable periodical which the Union List of Serials names Suomalaisen Tiedeakatemian Toimituksia, but which the Revue Semestrielle designates—equally correctly—as Annales Academiae Scientiarum Fennicae. One who searches for a volume of this rather scarce journal wishes that there were agreement on nomenclature.