THE MAY MEETING IN NEW YORK

The two hundred fifty-fifth regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, May 7, 1927, extending through the usual morning and afternoon sessions. Attendance included the following eighty members of the Society:

Alexander, Archibald, W. L. Ayres, F. W. Beal, A. A. Bennett, Blichfeldt, B. H. Camp, G. A. Campbell, Carmichael, W. B. Carver, Curry, Douglas, Edmondson, Farnum, Fite, D. A. Flanders, Fort, Philip Franklin, Gehman, Gill, Gronwall, C. C. Grove, Haskins, Hedlund, Himwich, Hollcroft, Ingraham, Joffe, W. A. Johnson, O. D. Kellogg, Kholodovsky, Kline, Koopman, Kormes, Langford, Langman, Lefschetz, Littauer, Litzinger, Lotka, MacColl, W. A. Manning, Michal, H. H. Mitchell, L. T. Moore, Mullins, Neelley, K. E. O'Brien, F. W. Owens, H. B. Owens, Paradiso, Pell-Wheller, Pfeiffer, Phalen, Pierpont, Post, R. G. Putnam, Raynor, R. G. D. Richardson, Ritt, Ruger, Rutt, Seely, Siceloff, Simons, Smail, Virgil Snyder, M. H. Stone, Teach, J. M. Thomas, Tracey, Veblen, Wedderburn, Weida, Weisner, H. S. White, Whittemore, Wiener, J. W. Young, Margaret M. Young.

The Secretary announced the election of the following eighteen persons to membership in the Society:

Miss Margaret Charlotte Amig, St. Catherine's School, Richmond;
Dr. Walter Bartky, University of Chicago;
Mr. John Montgomery Clarkson, Duke University;
Mr. Hubert Leron Clary, radio engineer, Milwaukee;
Mr. Thomas Freeman Cope, Adelbert College, Western Reserve University;
Mr. Henry Leslie Garabedian, University of Rochester;
Mr. José Treviño García, civil engineer, Parras, Mexico;
Mr. John Jay Gergen, Rice Institute;
Dr. Lulu Hofmann, Springfield, Ohio;
Professor Robert E. Hundley, University of Cincinnati;
Mr. Vern James, Stanford University;
Professor Lida Belle May, Kidd-Key College;
Professor Harry L. Miller, University of Cincinnati;
Mr. Bernard Preston, certified public accountant, New York City;
Mr. Ralph Grafton Sanger, University of Wisconsin;
Professor Harold Ward Sibert, University of Cincinnati;
Dr. Waldemar Joseph Trjitzinsky, University of Texas;
Professor Charles Newman Wunder, University of Mississippi.

The meeting of the Board of Trustees took place in the Murray Hill Hotel on the evening of May 6, and that of the

Council on May 7 at 9:30 A.M., in Columbia University.

It was voted by the Trustees that all funds bequeathed to the Society should be added to endowment unless specifically designated for other purposes.

Professor E. W. Brown was invited to give the fifth Josiah Willard Gibbs Lecture in connection with the Annual Meeting to be held at Nashville.

The following appointments were announced: to represent the Society at the Bicentennial of the American Philosophical Society in Philadelphia, April 27–29, Professors G. D. Birkhoff, L. E. Dickson, and H. B. Fine; to represent the Society at the Centennial of Lindenwood College, St. Charles, Missouri, Professor W. H. Roever.

It was announced that Professor R. L. Moore has accepted the invitation to give a Colloquium at the Summer Meeting of 1929 in Boulder, and that Professor J. H. M. Wedderburn has been appointed an associate editor of the Bulletin.

An invitation was received from Amherst College to hold a mathematical conference of a few weeks duration about the time of the Summer Meeting which is to be held there in 1928. Brown University presented an invitation for the Summer Meeting of 1930 or 1931.

The following committee was appointed to nominate to the Council a list of Officers and Members of the Council for 1928: Professors D. R. Curtiss (chairman), W. B. Ford, D. C. Gillespie, E. R. Hedrick, and W. H. Roever.

Professor Hollcroft presided at the beginning of the morning session, relieved by Ex-President Veblen and President Snyder. Vice-President Kellogg and President Snyder presided at the afternoon session.

At the request of the Program Committee, Professor J. R. Kline delivered an address, at the beginning of the afternoon session, entitled *Separation theorems and their relation to recent developments in analysis situs.* Titles and abstracts of the other papers read at the meeting follow below. The papers of Alexander, Bray, Dodd, Douglas (third and fourth papers), Flanders, Garver, Haskell, Kasner, McVey and Babb, Maria, Merriman, Murnaghan, Musselman, Richmond, Serghiesco, Weisner (second paper), and Whyburn were read by title. Professor Jules Drach was introduced by Professor R. G. D. Richardson, Mr. McVey by Professor Babb, Mr. Serghiesco by Professor Ritt, and Professor Struik by Professor Wiener.

1. Dr. Louis Weisner (National Research Fellow): *On m-dimensional cross ratios.*

Regarding the projective invariance of the cross ratio of four collinear points as its essential property, many writers have proposed unsatisfactory generalizations. The characteristic property of the cross ratio is not its invariance under projection, as an arbitrary function of the cross ratio also enjoys this property, but is that expressed by the following theorem: A necessary and sufficient condition that two sets of four collinear points be projective is that their respective cross ratios be equal. In generalizing to m-space, we seek functions which, in a similar way, provide conditions that two sets of $m+3$ points in m-space be projective. Such functions (m of them in m-space) have been proposed in Veblen and Young's *Projective Geometry*, vol. 2, p. 55. It is shown in the present paper that these cross ratios have properties which are generalizations of the known properties of the classical cross ratio. Applications are made to Coble's associated point sets.

2. Dr. Louis Weisner: *A new reciprocity law in invariant theory.*

The reciprocity law is a result of the observation that from an invariant I of degree d of an m-ary form of order n, expressed as the product of symbolic determinants of order m, we may form an invariant I' of degree d of a k-ary form of order kn/m, where $k=d-m$, by replacing each determinant D of I by the determinant of order k consisting of those symbols in I which are not contained in D. By the same rule I is obtained from I'. For example, from the invariant $(ab)^2(ac)^2(de)^2(bd)(be)(cd)(ce)$ of the binary quartic we form the invariant $(cde)^2(bde)^2(abc)^2(ace)(acd)(abe)(abd)$ of the ternary sextic. The law breaks down if one and only one of the two corresponding invariants vanishes. That this may actually occur is shown by an example. However, if one or two corresponding *absolute* invariants has the value 0, ∞, or $0/0$, the other also has this value. The reciprocity law is therefore valid for absolute invariants without exception.

3. Dr. Louis Weisner: *The functional equation defining diophantine automorphisms.*

This paper will appear in full in an early issue of this Bulletin.

4. Dr. L. T. Moore and Dr. J. H. Neelley: *Rational tacnodal and oscnodal quartic curves considered as plane sections of certain quartic surfaces.*

This paper develops a system of invariants, which are functions of the coefficients of the cutting plane, for both the tacnodal and the oscnodal rational quartic curves. We also distinguish between the two types of tacnodes by means of an invariant.

5. Dr. L. T. Moore: *Note on the nodes of the rational plane quartic.*

The nature of the nodes of the rational quartic, given parametrically or as a section of the Steiner quartic surface, is determined from the invariants of the curve.

6. Professor T. R. Hollcroft: *The generalized Hessian.*

The definitions of the Hessian, Steinerian, and Cayleyan were extended by Salmon to include the covariant curves similarly related to the systems of first, second, third, etc., polars of an algebraic curve, but he did little more than define them. Any generalized Steinerian is also a generalized Hessian. The ordinary Steinerian now appears as a Hessian with respect to the net of polar conics. All generalized Hessians have distinct nodes and cusps except the first Hessian. The characteristics of the generalized Hessian are found for a non-singular basis curve and for a basis curve with simple or compound singularities. The order and other characteristics of certain generalized Hessians are reduced by singularities of the basis curve. For every curve of odd order, a certain generalized Hessian coincides with its corresponding Steinerian. This curve is given the name mid-Hessian. The Hessian of the cubic is the simplest example. A property of the mid-Hessian is that it can never have more than one singular point other than distinct nodes and cusps and this singularity can not be compound.

7. Mr. A. J. Lotka: *The progeny of a population element.*

In connection with population statistics as well as with the mathematical theory of evolution, it is of interest to know how the successive generations descended from an initial population element are distributed in time. Given a frequency distibution in any one generation, the nth, say, it is shown that the kth seminvariant of the frequency distribution of the $(n+1)$th generation is obtained by adding to the kth seminvariant of the nth generation the kth seminvariant of the function $f(a)=s(a)\beta(a)$, where $s(a)$ is the fraction surviving to age a, out of a batch of births, while $\beta(a)$ is the frequency of reproduction at age a. (The discussion is restricted to the case in which $s(a)$ and $\beta(a)$ are independent of the time, and are functions of the age a only. As corollaries we have the following: (1) the frequency quency distribution of births in advanced generations is essentially independent of the frequency distribution in the initial population element; (2) the frequency distribution of births ultimately approaches more and more nearly to the "normal" or Gaussian distribution. Formulas are also developed for (a) the frequency distribution expressing the relative contribution of coexisting "successive" generations, to the total birth rate at a given time, and (b) the total birth rate at any instant.

8. Professor Philip Franklin: *A qualitative definition of the sub and super harmonic functions.*

An integral characterization of functions satisfying Poisson's equation with non-negative density has recently been given by F. Riesz (Acta Mathematica, vol. 48, pp. 329 ff.) which identifies them with the super harmonic functions studied in other connections. In the present paper we give a set of postulates characterizing these functions which are of a more qualitative nature, in that they involve neither derivatives nor integrals. They are analogous to a set previously given for the harmonic functions themselves (this Bulletin, vol. 30, pp. 41 ff.).

9. Dr. Jesse Douglas (National Research Fellow): *The most general geometry of paths.*

A system of paths in an n-dimensional space S_n is defined as any analytic system of curves in S_n such that a unique curve passes through any given two points, or through any given point in any given direction. According as the paths are considered in a fixed parameterization or independent of any parameterization, we speak of an *affine* or *projective* space of paths. The principal theorems of this paper are the following: (1) Any affine space of paths is representable by $d^2x^i/dt^2 = H_2^i(x, p)$, $(p = dx/dt)$, where $H_2^i(x\ p)$ denotes a function homogeneous of the second degree in p. (2) Any projective space of paths is representable by $d^2x^i/dt^2 = H_2^i(x, p) + p^iH_1(x\ p)$, where $H_2^i(x, p)$ are fixed functions of their kind, and $H_1(x, p)$ denotes an arbitrary homogeneous function of the first degree in p, whose variation affects only the parameterization of the paths, not the paths themselves. These results and their proofs will appear in the Proceedings of the National Academy, and an elaboration developing the basis of the geometry of a general system of paths in an early number of the Annals of Mathematics.

10. Dr. Jesse Douglas: *A method of numerical solution of the problem of Plateau.*

By replacing the partial derivatives in the equation of minimal surfaces $(1+q^2)r - 2pqs + (1+p^2)t = 0$ by their finite difference approximations, a formula is derived giving the value of a solution at the center of a small square in terms of the values at the midpoints of the sides and the vertices. A numerical solution of the problem of Plateau is based on the use of this formula in connection with successively finer and finer nets of square meshes constructed over the xy-plane. Applied to a chosen contour on the catenoid $z = \cosh^{-1}(x^2+y^2)^{1/2}$, this method gives numerical results agreeing to four decimal places with a table of hyperbolic functions. The method is then applied to a contour where the solution is not known in advance.

11. Dr. Jesse Douglas: *A method of numerical solution of the problem of Dirichlet for a square.*

The value at the center of a small square, side $2h$, in the xy-plane, of any solution of Laplace's equation $\partial^2z/\partial x^2 + \partial^2z/\partial y^2 = 0$ is, up to infinitesimals of the order h^4, the average of the values at the midpoints of the sides;

also the average of the values at the vertices. It is found that if the former values are weighted four times as heavily as the latter, the resulting average gives the value at the center up to infinitesimals of the order of h^6. On the basis of this relation, a numerical solution of different particular cases of the problem of Dirichlet for a square is effected by a successive approximation procedure involving the use of a sequence of nets with square meshes. It is expected in a future paper to apply this method to numerical computation of the function $\wp(u)$ for which $g_2=1$, $g_3=0$.

12. Dr. Jesse Douglas: *Contact transformations of 3-space which convert a system of paths into a system of paths.*

The following theorem is proved: Given a proper contact transformation Γ(i.e., one not reducing to a point transformation) of a 3-space S_3 into another S_3'; if there exists a system of paths in S_3 (in the sense of our first paper, above) which is converted by Γ into a system of paths in S_3', then both systems of paths must be linear (equivalent by point transformation to the straight lines of a projective space), and Γ must have the form PDP_1, where P, P_1 are any point transformations and D is a dualistic transformation.

13. Professor James Pierpont: *On the geometry whose absolute is a ruled quadric.*

The equation of a non-degenerate quadric has one of the three following forms: $x_1^2+x_2^2+x_3^2+x_4^2=0$, $x_1^2+x_2^2+x_3^2-x_4^2=0$,$x_1^2+x_2^2-x_3^2-x_4^2=0$. Each of these taken as the absolute defines a geometry; the first gives elliptic, the second gives hyperbolic geometry. The last is a ruled quadric, and the question arises as to what kind of a geometry this defines. Poincaré (Bulletin de la Société de France, vol. 15(1886), p. 206) has stated a few properties for the plane; the author has seen no other reference. The present paper undertakes a study of this space.

14. Dr. C. H. Langford: *On inductive relations.*

Difficulties, connected with the occurrence of reflexive fallacies, appear when we attempt to give straighforward definitions of inductive series. These difficulties arise inevitably from the fact that, if a series is to be inductive, in the ordinary sense, properties of all orders must be transmitted in the series, and it would seem that no proposition, nor any finite set of propositions, can assert that properties of all orders are transmitted. In this paper, it is held that, although propositions in extension are strictly subject to differences of type, propositions in intension are not; and that propositions in intension, modal propositions, fall outside the heierarchy, and may, in an important sense, be about any proposition. This distinction of modal and material propositions leads to an analogous distinction of modal and material properties; and modal properties are important in definitions of inductive series.

15. Dr. B. O. Koopman: *On multiple-valued functions of several complex variables.*

We consider the infinitely multiple-valued monogenic function $f(z_1, \cdots, z_n)$, every branch of which is meromorphic in a finitely multiply-connected region R of the complex $z_1 \cdots z_n$-space. Evidently $f(z_1, \cdots, z_n)$ admits an infinite monodromy group. By the rank r of $f(z_1, \cdots, z_n)$ shall be meant the maximum number of functionally independent branches at any point. When $f(z_1, \cdots, z_n)$ is given, r is determined, and $1 \leqq r \leqq n$. We show that there is a $2r$-dimensional region Ω on a complex $u_1 \cdots u_r$-space which admits a group of (1-1) analytic automorphisms, isomorphic with the monodromy group, and having the same effect on the branches. In general, $r = n$. The case $r < n$ arises if and only if $f(z_1, \cdots, z_n)$ is an integral of $n-r$ independent equations of the form $Z_1(\partial f/\partial x_1) + \cdots + Z_n(\partial f/\partial z_n) = 0$, the Z's being single-valued meromorphic functions of $(z_1, \cdots, z_n)$ in R. When $r = n-1$, and when $f(z_1, \cdots, z_n)$ is real, Ω and its group of automorphisms forms an immediate generalization of the ring-transformation (surface of section) for the dynamical system $dz_1/Z_1 = \cdots = dz_n/Z_n$. Other special cases are furnished by the inverses of certain automorphic and Cremona functions.

16. Dr. B. O. Koopman: *The Riemann multiple-space and algebroid functions.*

The object of this paper is two-fold: first, we define the notion of the Riemann surface in its extension to the case of several variables from the point of view of abstract sets; the desideratum is that the result shall be sufficiently general for the purposes of the theory of functions, without having unnecessary generality. The second and more extensive part of the paper studies the result, from the point of view of analysis situs, in the case of functions defined by algebroid and algebraic equations. The methods used are those of the theory of analytic factorization (see Osgood, *Funktionentheorie*, vol. 2, Chapter 2), and of combinatorial analysis (see Veblen, The Cambridge Colloquium, Part II). After dividing the Riemann multiple-space into a complex of analytic cells, it is shown that, in the case of algebraic functions, the result is a generalized manifold (Veblen, loc. cit., p. 92), while the branch points and non-spherical points form circuits of cells of the complex, of certain specified sorts. Corresponding theorems may be stated in the case of algebroid functions.

17. Mr. S. Serghiesco: *On an integral giving the number of the common roots of a system of simultaneous equations.*

First suppose that two equations $f=0$, $\phi=0$ are given. We assume that the conditions that Picard's well known formula giving the common roots of this system shall reduce to a line integral are satisfied. In the present paper a new and general integral is obtained, giving the same number of roots by making use of the differential invariants of $f+\lambda\phi=0$. This integral is then generalized, for the case of n equations $f_1=0, f_2=0, \cdots, f_n=0$, to an integral of order $n-1$, by means of the differential invariants of $f_1+\lambda f_2 + \mu f_3 + \cdots = 0$. A verification of Picard's results for the case of polynomials is also given through this new formula.

18. Mr. R. N. Haskell: *A note on Stieltjes integrals.*

Given $h(P)$, a continuous function of the point P, and $F(w)$, a bounded additive function of regular curves w, and a sequence $\{\delta_n\}$ of positive numbers converging to zero; if the fundamental region Δ be divided into finite number K_n of simply connected regions bounded by regular curves $\{W_n^i\}$, with diam.$(W_n^i) < \delta_n$, $(i=1, 2, \cdots, K_n)$, then the Riemann sum $\sum_i h(P_n^i)F(W_n^i)$ approaches a unique limit $\int_\Delta h(P)dF(w)$ as $n\rightarrow\infty$, independently of the system of curves W_n^i by which it is calculated. This limit is therefore equal to the Stieltjes integral $\int h(P)dF(s)$ as usually defined by means of segments, s.

19. Dr. G. M. Merriman (National Research Fellow): *Concerning the summability of double series of a certain type.*

The discussion in the present paper is pointed to a general theorem containing, for example, the result that if $a_{m,n}$ is the general term of a double series summable (C, r, s), then $a_{m,n}m^{-h}n^{-k}$ is the general term of a double series which is summable $(C, r-h, s-k)$. The discussion is carried out through generalizations to double series of the Rieszian summation means, here defined for the first time; and it includes, necessarily, a proof of the equivalence, as a means of summation, of certain of these Rieszian means to the double Cesàro means. A by-product is the important theorem that a series summable (C, r, s) is also summable (C, r', s'), $r'>r$, $s'>s$; this result has been proved before only for integral values of r, r', s, s'.

20. Professor F. D. Murnaghan: *On the degree of arbitrariness of the wave equation in the new quantum mechanics.*

For a dynamical system with fixed constraints, moving in a stationary conservative force field, the action, $W=-Et+S(q)$, satisfies the Hamilton equation $\partial W/\partial t+H(q, \partial W/\partial q)=0$. Here q denotes the positional coordinates of the system, and E the constant total energy, both kinetic and potential, of the system. The partial differential equation of the family of moving surfaces $F(q, t)=0$, on each of which W takes a constant value, is found by putting $F=f(W)$, whence $\partial W/\partial q=(\partial W/\partial t)((\partial F/\partial q)/(\partial F/\partial t))$ $=-E((\partial F/\partial q)/(\partial F/\partial t))$, and $H(q, -E(\partial F/\partial q)/(\partial F/\partial t))$. For a moving particle we get the familiar equation $(\partial F/\partial x)^2+(\partial F/\partial y)^2+(\partial F/\partial z)^2$ $=(2M(E-V)/E^2)(\partial F/\partial t)^2$. The wave equation is to have the surfaces $F=0$ for its wave fronts, i.e. characteristics. It may be written $g_{11}\partial^2\psi/\partial x^2$ $+g_{22}\partial^2\psi/\partial y^2+g_{33}\partial^2\psi/\partial z^2=g_{44}\partial^2\psi/\partial t^2+\phi$ where g_{11}, for example, is any function of $(x, y, z, t, \psi, \partial\psi/\partial x, \partial\psi/\partial y, \partial\psi/\partial z, \partial\psi/\partial t)$ which reduces to 1 when ψ and the first-order derivatives are zero. ϕ is any function of the same arguments reducing to zero when ψ and its first-order derivatives vanish. The introduction through ϕ of terms in the first order derivative of ψ is interesting in connection with the theory of a magnetic field.

21. Professor E. L. Dodd: *The probability law for the intensity of a trial period, with data subject to the Gaussian law.*

This paper will appear in full in an early issue of this Bulletin.

22. Dr. Raymond Garver: *A series solution for certain rational cubics.*

Any rational cubic, with minor exceptions, can be transformed rationally into an equation of the form $x^3+A(x+1)=0$. By means of certain substitutions and an inversion we obtain an expression for a real root of this equation as a power series in $1/(A+m)$. The series for $m=4$ is studied.

23. Dr. Raymond Garver: *Arccotangent relations and expressions for* π.

A number of arccotangent relations are developed (some of which the author has not seen before); these lead to several new arctangent expressions for π

24. Dr. Raymond Garver: *A type of function with k discontinuities.*

An analytic expression is given for a type of function which is, in general, discontinuous at k points, where k is any positive integer.

25. Professor J. R. Musselman: *On an imprimitive group of order* 5184.

In an earlier paper, read before the Society December 29, 1924, the author discussed a certain configuration in three-dimensional space. In the present paper he considers the imprimitive group of order 5184 under which the configuration is invariant. Associated with the configuration are eight Eckhardt cubic surfaces and twenty-seven sets of desmic tetrahedra. A similar problem in the plane leads to an imprimitive group of order 216 associated with four Hesse configurations. Both papers will appear together in the American Journal of Mathematics.

26. Dr. G. T. Whyburn: *Concerning irreducible cuttings of a continuum.*

A subset K of a continuum M is a *cutting* of M provided $M-K$ is not connected; K is an *irreducible cutting* of M if no proper subset of K cuts M; K is a *cutting of M between the points A and B of M* provided $M-K$ is the sum of two mutually separated sets M_a and M_b containing A and B respectively. Let M denote any continuum, A and B any two points of M, and K an *irreducible cutting of M between A and B*, (if one exists). It is shown in this paper that (1) if $M-K=M_a+M_b$, as above, then M_a+K and M_b+K are continua; (2) if M is indecomposable, no set K can exist; (3) if M is a continuous curve, then every cutting of M between A and B contains a set K; (4) if every subcontinuum of M is a continuous curve, then every cutting of M contains an irreducible cutting of M; and (5) if M is a bounded plane continuum, and H is an irreducible cutting of M which has more than two maximal connected subsets, then $M-H$ is the sum of two mutually separated *connected* point sets.

27. Dr. G. T. Whyburn: *On the separation of indecomposable continua and other continua.*

In this paper it is shown that if M is any indecomposable continuum and $[A_i]$ is any countable collection of mutually exclusive subcontinua of M, then no subset of $\sum A_i$ disconnects M. It is also shown that if M is any continuum whatever, X and Y are any two points of M, and K denotes the set of all those points of M which separate X from Y in M, and H is any connected subset of K, then (1) every point of H, save possibly two, is a cut point of H; (2) if H is closed, it is a simple continuous arc; (3) if A is a cut point of H, then $H-A$ is the sum of two mutually separated connected point sets; (4) if A and B are any two points of H, then H contains one and only one connected subset which is irreducibly connected between A and B; (5) if H has two non-cut points A and B, then H is irreducibly connected between A and B; and conversely, if H is irreducibly connected between some two of its points, these points are the non-cut points of H.

28. Professor H. F. Blichfeldt: *The reduction to six terms of the general quadratic differential form in four variables.*

This paper shows that the general quadratic differential form in four variables, $\sum\sum \alpha_{ij}dx_idx_j$, $(i, j=1, 2, 3, 4)$, of non-vanishing determinant, can, by a suitable choice of variables, $y_1, \cdots, y_4$, be reduced to the form $\beta_{11}dy_1^2+2\beta_{12}dy_1dy_2+\beta_{22}dy_2^2+2'\beta_{14}dy_1+\beta_{24}dy_2+\beta_{34}dy_3)dy_4$. Corresponding reductions are given for the general quadratic form in n variables.

29. Professor W. A. Manning: *The primitive groups of class fourteen in which the positive subgroup is of class greater than fifteen.*

All the primitive groups of class 14 contain negative permutations. Those in which the positive subgroup is of class 15 are best studied in connection with the primitive groups of class 15. The author announces that there are but four primitive groups of class 14 in which the positive subgroup is of class greater than 15. There are (1) the simply transitive primitive group according to which the symmetric group of degree 9 permutes its 36 transpositions; (2) the simply transitive primitive group of degree 49 and order $3(7!)^2$ isomorphic to the group generated by (ab), $(bcdefg)$, $(\alpha\beta)$, $(\beta\gamma\delta\epsilon\zeta\eta)$, and $(a\alpha)(b\beta)(c\gamma)(d\delta)(e\epsilon)(f\zeta)(g\eta)$; (3) the triply transitive group of degree 22 and order $22\cdot 21\cdot 20\cdot 96$ which occurs in Mathieu's quintiply transitive group of degree 24 as a transitive constituent of the largest subgroup in which the subgroup that leaves two letters fixed is invariant; (4) the doubly transitive subgroup of (3) of degree 21 and order $21\cdot 20\cdot 96$.

30. Professor Edward Kasner: *Irregular differential invariants. II: The group of all analytic transformations.*

This paper is a continuation of an earlier paper by the author, read at the February, 1927, meeting of the Society; it gives a qualitative classification of irregular analytic elements (represented by fractional power series). Some species, like $(p=2, q=3)$, have no invariants; others, like $(p=5, q=6)$, have invariants.

31. Professors D. J. Struik and Norbert Wiener: *The quantum theory as a corollary of relativity.*

In a general manifold of four dimensions, we assume a linear partial differential equation of the second order and the hyperbolic type. By means of the appropriate invariant theory, we cast it into a unique normalized form, involving a tensor of the second order, a vector, and an invariant scalar, representing respectively the gravitational field, the electromagnetic field, and a term connected with the quantum theory. On the assumption of the Einstein field equations for the gravitational field, our equation assumes a form which has already been given for the Schrödinger wave equation of quantum mechanics.

32. Professor G. A. Pfeiffer: *Certain sequences of curves which approach a rectifiable boundary from within.*

In this paper there is established the fact that if the boundary of a region G(a set of inner points) is a simple rectifiable closed curve C, then there exists a sequence of simple rectifiable closed curves, C_i, such that (1) every curve C_i is in G, (2) every point of G is in the interiors of all but a finite number of the curves C_i, and (3) if l_i is the length of C_i, then the length of C is the limit of the sequence $\{l_i\}$. The above theorem is obtained by means as elementary as the matter seems to permit. It is then shown that the curves of the sequence $\{C_i\}$ may be taken as equipotential lines of the region G.

33. Mr. W. L. Ayres: *Concerning the arc curves and basic sets of a continuous curve.*

If K is a subset of a plane continuous curve M, the arc curve of K with respect to M (denoted by $M(K)$) is the set of all arcs of M whose end points belong to K. Any subset of M whose arc curve is M itself is called a basic set of M. A large number of the properties of arc curves are obtained in this paper. Probably the most important are the following: (1) every point which is a limit point of $M(K)$ but not a point of $M(k)$ is a limit point of K; (2) $M(K)$ is arc-wise connected im kleinen; (3) whenever K is closed, $M(K)$ is a continuous curve; (4) every interior point of an arc of M whose end points are points or limit points of $M(K)$ belongs to $M(K)$. Necessary and sufficient conditions are given in order that a subset of M be a basic set of M and in order that a basic set of M be irreducible.

34. Mr. W. L. Ayres: *On the structure of a plane continuous curve.*

In this paper the limiting arc curve of a continuous curve M at a point is defined, and the following statements are proved: (1) if P is a point of M lying on no simple closed curve of M and α is any arc of M containing P, then P is a limit point of the cut points of M which lie on α; (2) the set of all simple closed curves of M containing a given point P is a continuous curve; (3) if P is a point of M, the limiting arc curve of M at P(a) exists and consists of the single point P, (b) exists and is the set of all simple closed curves of M containing P, (c) does not exist, if and only if (a) P lies on no simple closed curve of M, (b) P is a non-cut point of M and lies on some simple closed curve of M, (c) P is a cut point of M and lies on a simple closed curve of M; (4) a continuous curve M is acyclic if and only if the limiting arc curve of M at P is P for every point P of M; (5) a continuous curve M is cyclically connected if and only if it contains a point P such that the limiting arc curve of M at P is the set M itself.

35. Mr. N. E. Rutt: *Concerning the cut points of a continuous curve when the arc curve, ab, contains exactly n independent arcs.*

In this paper the author proves that if a and b are two distinct points of a continuous curve E, and if there exists a number n such that in E from a to b there are n independent arcs and only n, then there are in E n points $P_1, P_2, \cdots, P_n$ such that $E-\sum_{i=1}^{n} P_i$ is disconnected and a and b are not in the same connected subset.

36. Mr. D. A. Flanders: *The double elliptic geometry in terms of point, order, and congruence.*

The set of axioms presented in this paper possessess two properties of particular interest. The first may be termed "dimensional categoricity," by which is meant that every proposition in space of a given dimensionality, n, is demonstrable in terms of axioms whose hypotheses predicate the existence of not more than n dimensions. The second may be described by saying that the space is characterized by its one-dimensional axioms; i.e., in order to extend the space beyond the first dimension, the only new axioms required are existence and closure axioms.

37. Dr. D. E. Richmond (National Research Fellow): *Number relations between types of extremals joining a pair of points.*

Let A and B be a pair of points in an extremal-convex region for a calculus of variations problem. An extremal arc joining A to B may be assigned a type number equal to the number of points on the arc conjugate to A. Existence theorems are obtained for certain relations between the numbers of extremals of different types which join A to B, by an appropriate choice of integrand and by application of envelope theory.

38. Professor H. E. Bray: *Is a function of écart fini necessarily of limited variation?*

In Liouville's Journal, (4), vol. 8 (1892), p. 166, Hadamard raised the question as to whether a continuous function of écart fini is necessarily of limited variation. In this note it is proved that the function $x \sin (1/x)$ is of écart fini. The answer is therefore in the negative, for the function in question is also continuous, but is not of limited variation in the interval $0 \leqq x \leqq 2\pi$.

39. Dr. A. J. Maria (National Research Fellow): *Derivatives of functions of plurisegments.*

The author obtains theorems concerning the derivative of a function of plurisegments with respect to an absolutely additive function of point sets. Use is made of the results contained in an article by the author which is to appear in the Annals of Mathematics.

40. Dr. A. J. Maria: *Derivatives of functions of several variables.*

In this paper the author shows that a function of bounded variation $f(x, y)$, defined in the rectangle R $(0 \leqq x \leqq a,\ 0 \leqq y \leqq b)$ can be represented as the sum of two functions $s(x, y)$ and $a(x, y)$. The function $s(x, y)$ is such that $\partial^2 s/\partial x\partial y$ is zero nearly everywhere in R, and $a(x, y)$ is equal to $\int_0^x\int_0^y(\partial^2 f/\partial x\partial y)dxdy$. The theorem rests on the theory of functions of plurisegments developed by the author in a previous article (Transactions of this Society, vol. 28 (1926), pp. 448–471).

41. Mr. P. McVey and Professor M. J. Babb: *On $(n$-$k)$ numbers.*

A number n the sum of whose factors is nk is called an $(n$-$k)$ number. A technique is developed for even $(n$-$k)$ numbers, and it is shown that no odd $(n$-$k)$ numbers exist.

42. Professor J. W. Alexander: *Topological invariants of knotted curves in 3-space.*

In this paper, certain topological invariants of knotted curves in 3-space are derived. The principal invariants are in the form of polynomials with integer coefficients.

43. Professor Jules Drach: *Integration by linear systems of equations of the type $r+f(s, t)=0$.*

This paper gives some unexpected results on the general integration by linear systems of equations (1): $r+f(s, t)=0$, where $dz=pdx+qdy$, $dp=rdx+sdy$, $dq=sdx+tdy$. If m_1, m_2 are the roots of $m^2-m(\partial f/\partial s)+\partial f/\partial t=0$, we have for the *characteristics* of Cauchy two integrable combinations: $ds+m_2dt=Ad\alpha$, $ds+m_1dt=Bd\beta$. With the new variables α, β, equation (1) can be replaced by (2): $F(Z)=(\partial^2 Z/\partial\alpha\partial\beta)\div(4(\partial Z/\partial\alpha)\cdot(\partial Z/\partial\beta))=\lambda^2$, and $\lambda(\alpha, \beta)$ is such that r and t are two solutions of (2). Equation (2) may be transformed by $dZ=u^2d\alpha+v^2d\beta$ into a linear system (3): $\partial u/\partial\beta=\lambda v$, $\partial v/\partial\alpha=\lambda u$. For each form of $\lambda(\alpha, \beta)$, we have, for *two arbitrary solutions* of (3), $dr=u_2{}^2d\alpha+v_2{}^2d\beta$, $ds=u_1u_2d\alpha+v_1v_2d\beta$, $dt=u_1{}^2d\alpha+v_1{}^2d\beta$, and we may then obtain equation (1). With the *general solution* u, v of (3) we have $u_1y+u_2x=u$, $v_1y+v_2x=v$, and $z=\frac{1}{2}Z+\frac{1}{2}(rx^2+2sxy+ty^2)-(xP+yQ)$, where $dP=uu_2d\alpha+vv_2d\beta$, $dQ=uu_1d\alpha+vv_1d\beta$, $dZ=u^2d\alpha+v^2d\beta$; these equations give x, y, z as functions of α, β. By means of (2) and a transformation due to Goursat, we may obtain, in a regular way, all the equations (1) that may be integrated by quadratures only.

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Secretary.