THE APRIL MEETING IN CHICAGO

The twenty-seventh Western meeting of the Society was held at the University of Chicago on Friday and Saturday, April 15 and 16, 1927. The attendance at this meeting included about one hundred and twenty persons, among whom were the following seventy-seven members of the Society:

B. M. Armstrong, Frank Ayres, R. P. Baker, G. A. Bliss, Blumberg, Brahana, Brand, O. E. Brown, C. C. Camp, Chittenden, Conkwright, Cope, Curtiss, H. T. Davis, Denton, Dickson, Dresden, Emch, Frink, C. A. Garabedian, Raymond Garver, Gouwens, Griffiths, Harkin, Harshbarger, W. L. Hart, Hickson, Hodge, Hofmann, L. A. Hopkins, Dunham Jackson, R. L. Jackson, B. W. Jones, Krathwohl, LaPaz, Lytle, McFarlan, MacDuffee, W. D. MacMillan, Marquis, T. E. Mason, E. H. Moore, E. J. Moulton, Mullings, Palmer, Parkinson, Pettit, Rainich, C. J. Rees, C. E. Rhodes, D. P. Richardson, H. L. Rietz, P. G. Robinson, Roos, Roth, Schottenfels, Sharpe, J. B. Shaw, Sherer, Shohat, Simmons, W. G. Simon, Sisam, Slaught, A. W. Smith, Virgil Snyder, J. H. Taylor, E. L. Thompson, J. S. Turner, Van Vleck, Vass, Wahlin, L. E. Ward, Warren Weaver, John Williamson, F. E. Wood, J. W. A. Young.

On Friday afternoon Professor E. W. Chittenden gave the symposium address entitled *Some aspects of general topology*. This address will appear in full in an early issue of this Bulletin.

On Friday evening members and guests of the Society gathered at a dinner in the Del Prado Hotel. President Snyder presided and called on Professors Coble, Bliss, Slaught and Dresden, who spoke of various recent developments in the work of the Society.

The papers listed below were presented on Friday and Saturday forenoons. The first session was divided into two sections: the first, on Geometry and Point Sets, at which President Snyder presided, included the papers numbered 1 to and including 11, and 48; before the second section, on Algebra, Theory of Numbers and Applied Mathematics, presided over by Professor D. R. Curtiss, were presented the papers numbered 12 to and including 20, 45 and 50. The remaining papers were read at a long session on Saturday forenoon, presided over by Professors Rietz, Jackson and Bliss. The papers numbered 7, 8, 9, 10, 11, 18, 21, 23, 27, 34 to and including 42, 45 were read by title. Professor Uspensky was introduced to the Society by Professor Dresden, and Mr. Keller by Professor W. D. MacMillan.

1. Dr. H. R. Brahana: *Regular maps and their groups.*

In this paper we prove that to every group generated by two operators one of which is of order two, there corresponds a regular map. The symmetric groups, the alternating groups, the metacyclic groups and certain classes of their sub-groups, and the sub-groups modulo n of the modular group are in this class. There are just eight regular maps on a surface of genus two; of these the only one which does not contain two regions adjacent to each other along more than one edge is a map of 16 triangles.

2. Professor H. P. Pettit: *A special quartic cyclide.*

The special quartic cyclide is generated by a pencil of planes and a related system of spheres quadratic in the parameter λ. The axis of the pencil of planes is a double line on the quartic. The quartic is tangent to the envelope of the family of spheres along a sextic curve, which also lies on two cubic cyclides. Any sphere cuts the quartic cyclide in the spherocircle and a finite sphero-sextic which lies on a ruled cubic. The quartic cyclide may be generated in a triply infinite number of ways by a pencil of cubics and a projectively related pencil of concentric spheres. There are 108 circles on the surface not belonging to the infinite system of circular generators. The locus of the centers of the generators is a rational space quartic. There exist eight pairs of minimal lines on the quartic.

3. Professor H. P. Pettit: *A sextic cyclide.*

The sextic cyclide under consideration is generated by a pencil of spheres and a related family of spheres quadratic in the parameter λ. The spherocircle appears as a triple conic and the base circle of the pencil as a double conic on the sextic. The envelope of the quadratic family of spheres is a quartic cyclide, tangent to the sextic along a space sextic curve. Any sphere cuts the sextic in a sphero-sextic which lies on a ruled cubic. The locus of the centers of the circular generators is a rational septic curve. There exist 9 pairs of minimal lines on the sextic.

4. Professor Solomon Lefschetz: *Correspondences on algebraic curves.*

In the 1926 volume of the Transactions of this Society the author developed a general theory of continuous transformations of manifolds. The object of the present communication is to apply it to correspondences on algebraic curves. The situation being simpler than for general manifolds, many proofs can be greatly simplified. Much of the theory may be developed with a minimum use of analytical machinery.

5. Professor M. I. Logsdon: *A hypersurface in S_4 invariant under the general projective group of points on a line.*

Associating with the G_4 on the x-line given by $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ a point $P(a_0a_1a_2a_3a_4)$ in a space of four dimensions, the triply infinite group of transformations on x will transform P into a variety of 3 dimensions. It is shown that the order of this variety is six, that the system of these varieties of index 1 has as common elements (a) the rational (skew) quartic in S_4; (b) the developable consisting of the lines tangent to this quartic; (c) developables consisting respectively of planes and three-spaces osculating the quartic. Special cases and the dual situation are of interest since they offer a geometric interpretation of well known facts concerning the invariants of the biquadratic form.

6. Professor M. I. Logsdon: *Curves in r-space invariant under a net of homographies containing the identity.*

Enriques (Rendiconti dei Lincei, 1890) studied invariant loci under pencils of homographies in a space of n dimensions; Bonola (Rendiconti del Istituto Lombardo, 1908) studied invariant loci under nets of homographies in three-space when identity is an element of the net. The author, extending the study to nets of general homographies in r-space, finds: (1) to every such net containing the identity homography, there corresponds an invariant curve of order $r(r+1)/2$ and genus $r(r-1)/2$, on which there is an involution, g^1_{r+1}, of groups of $r+1$ points not cut out by hyperplanes; (2) to every curve in r-space of the above properties, there corresponds a pencil of homographies in the space for which the curve is invariant, and whose construction is given; (3) this net is unique. Thus the problem of normalization of a net of homographies depends on the number of invariants of the net. This number is the genus of the curve. A later paper will discuss normal types when identity is not present in the net.

7. Professor R. M. Mathews: *Desmic configurations on pencils of syzygetic cubics.*

The locus of the quadruples of poles of Caporali's fixed line (v) with respect to the cubics of a syzygetic pencil of parameter m is a point quartic Q which passes through the vertices of the four triangles of flex axes of the pencil Q cuts each cubic m in 12 points which form three desmic quadrangles, and (v) is the common satellite line with respect to m of their 16 lines of perspectivity. The 18 sides of these quadrangles cut Q again by threes in 12 points which form the associated desmic configuration on the corresponding cubic m in a derived syzygetic pencil, and the common satellite line for the 16 lines here is the locus of the pole of (v) with respect to the class cubics of the syzygetic pencil dual to pencil m. This line is also the contact tangent of (v) for that cubic of the dual pencil which touches (v). It cuts Q in the four points which are the poles of (v) with respect to the four triangles of the flex axes.

8. Professor R. L. Wilder: *Concerning a theorem of J. R. Kline.*

J. R. Kline proved (see this Bulletin, vol. 23 (1917), pp. 290–292) the following theorem: If, in euclidean space R_n of n dimensions ($n>1$), M is a domain and G_1, G_2, G_3, $\cdots$ is a countable infinity of nowhere dense closed point sets no one of which disconnects any domain, then $M-(G_1+G_2+G_3+\cdots)\times M$ is a non-vacuous arcwise connected point set. The present paper shows that the condition that the sets G_n should be nowhere dense may be replaced by the much weaker condition that no G_n is identical with R_n. The relations between the thus modified conditions imposed on G_n and the following conditions are then discussed, viz., (a) G_n is closed and punctiform, (b) G_n is closed and of dimension $\leqq n-2$ (in the Menger-Urysohn sense). Either conditions (a) or (b) may be substituted for the modified conditions on G_n and the above theorem is still true.

9. Professor R. L. Wilder: *Concerning zero-dimensional sets in the plane.*

Sierpinski has shown (Fundamenta Mathematicae, vol. 2 (1921), p. 89) that sets which are zero-dimensional in the Menger-Urysohn sense are identical with those sets that are homeomorphic with subsets of the irrational points of the straight line, and hence (Fréchet, Mathematische Annalen, vol. 68 (1910), p. 154) homeomorphic with subsets of the set I of points in the plane both of whose coordinates are irrational. The present paper considers the conditions under which a zero-dimensional set M in the plane is *isotopic* with a subset of I. It is shown that the zero-dimensionality of the set is not sufficient, and that a necessary and sufficient condition is that M be locally separated by simple closed curves. By introducing the notion of accessibility from all sides a characterization of zero-dimensional sets in the plane is obtained, as well as another necessary and sufficient condition for isotopism of a set with a subset of I dependent upon arcwise accessibility from all sides. As a corollary of these results, it is shown that every plane punctiform F_6 is isotopic with a subset of I.

10. Dr. G. T. Whyburn: *On the separation of plane connected point sets.*

The following results are established. (1) If A, B, and C are points of a bounded continuum M such that no pair of these points disconnects M but their sum does separate M, then $M-(A+B+C)$ is the sum of two mutually separated *connected* point sets. (2) If the connected set M is separated into mutually separated sets M_1 and M_2 by the omission of n of its connected subsets $A_1, A_2, \cdots, A_n$ then $M_i(i=1,2)+A_1+A_2+\cdots+A_n$ is the sum of at most n mutually separated connected point sets. Furthermore, if $M_1+A_1+A_2+\cdots+A_n$ is the sum of k such sets, then $M_2+A_1+A_2+\cdots+A_n$ is the sum of at most $n-k+1$ mutually separated connected point sets. (3) A bounded continuum M is disconnected by the omission of two of its non-cut points A and B if and only if there exist two complementary domains of M such that both of the points A and B are accessible from each of these domains. Interesting generalizations of (1) and (3) are established.

11. Dr. G. T. Whyburn: *Concerning certain subsets of continuous curves and of other continua.*

In this paper the author shows that if H denotes the sum of the boundaries of all the complementary domains of a continuous curve M, then (1) H is strongly connected if and only if the set of points $M-H$ does not separate the plane even in the weak sense; (2) if $M-H$ contains no continuum, then H is regular (connected im kleinen). Result (1) is true for any continuum M such that the diameters of its complementary domains may be arranged into a null-convergent sequence. If M is a continuous curve every subcontinuum of which is a continuous curve, then (a) the set H, above, is regular; (b) if R is any arcwise connected subset of M, it has property S and every limit point P of R is regularly accessible from R. If a continuum M is regular at every point of the boundary K of one of its complementary domains, then K is a continuous curve. Every connected subset of a "regular curve" (sense of K. Menger) is connected im kleinen.

12. Professor C. C. MacDuffee: *A correspondence between quadratic ideals and matrices.*

The correspondence discovered by Poincaré between the elements of a linear associative algebra and matrices of a certain type implies a p-to-1 correspondence between certain of these matrices and the principal ideals of a quadratic field where p is the number of units in the field. It is found that this correspondence is included in a p-to-1 correspondence between the matrices of a more extensive class and all ideals of the field. For every ideal class it is possible to choose two matrices B_1 and B_2, the first corresponding to a reduced ideal, such that the totality of matrices corresponding to ideals of the class is given by xB_1+yB_2 where x and y range independently over all rational integers, not both zero.

13. Professors Frank Morley and A. B. Coble: *Eliminants.*

Sylvester's dialytic method for eliminating one variable from two algebraic equations of any order was extended by Professor Morley at the Toronto Congress to cover the simultaneous elimination of two variables from three equations of the *same* order. This paper gives a method for simultaneous elimination which is much more comprehensive in scope.

14. Professor L. E. Dickson: *Simpler proofs of Waring's theorem on cubes, with various generalizations.*

Write C_n for the sum of the cubes of n undetermined integers $\geqq 0$. The following forms represent all positive integers: tx^3+C_8 for $1\leqq t\leqq 23$, $t\neq 20$; $tx^3+2y^3+C_7$ for $1\leqq t\leqq 34$, $t\neq 10, 15, 20, 25, 30$; $tx^3+3y^3+C_7$ for $1\leqq t\leqq 9$, $t\neq 5$. The paper will appear in the Transactions of this Society.

15. Professor L. E. Dickson: *Generalizations of Waring's theorem on fifth powers.*

Denote $ax^5+\cdots+kw^5$ by $(a,\cdots,k)$, and call $a+\cdots+k$ its weight. If a form of weight 37 represents all positive integers p, its order exceeds 7. The only two forms of order 8 which represent every $p \leqq 6500$ are (1 1 2 3 4 6 8, 12) and (1 1 2 3 4 5 7, 14). From these and two others we obtain by partitioning a coefficient into two parts 58 distinct forms of order 9 which represent all integers $\leqq 3137$. Certain types of order 9 are fully determined. R. C. Shook and K. C. Yang have investigated sixth and seventh powers.

16. Professor L. E. Dickson: *Extension of Waring's theorem on sixth powers.*

Every positive integer is represented by both $(1_{10}, 2_{90}, 3_{72}, 4_{18})$ and $(1_{178}, 8_{99})$, which give the coefficients of the sixth powers in the two forms. Here c_n denotes that n coefficients are equal to c.

17. Professor L. E. Dickson: *Every positive integer is a sum of* 17 *biquadrates and the doubles of* 10 *biquadrates.*

For $s=0, 1, \cdots, 10$, it is proved that every positive integer is a sum of s doubles of biquadrates and $37-2s$ biquadrates. The case $s=0$ gives the best complete result to date on single biquadrates. Papers 16 and 17 combined will appear in the American Journal of Mathematics.

18. Professor O. E. Glenn: *On the generalization of the algebra of lower number theory. Invariants of the cyclic transformations.* Preliminary communication.

A non-homogeneous form in e variables and with integral coefficients may have a modular period of $s=e+\alpha-1$ residues γ_i of an integral modulus n, such that

$$(1)\quad F(\gamma_1, \gamma_2, \cdots, \gamma_e) \equiv 0,\ F(\gamma_2, \gamma_3, \cdots, \gamma_e, \gamma_{e+1}) \equiv 0, \cdots, F(\gamma_\alpha, \cdots, \gamma_s) \equiv 0,\ F(\gamma_{\alpha+1}, \cdots, \gamma_s, \gamma_1) \equiv 0, \cdots, F(\gamma_s, \cdots, \gamma_{e-1}) \equiv 0 \pmod{n}.$$

The necessary and sufficient condition is obtained by forming the eliminant of (1) by Poisson's method as modified for congruences by Stieltjes. The result is a congruence in one γ, as $Q(\gamma_1) \equiv 0 \pmod{n}$. The period is the least s for which this congruence can be satisfied by a γ. If F is linear and homogeneous, Q is a cyclic determinant of order s and an invariant of F in reference to a general cyclic transformation. If $e=2$, Q is Euler's binomial $a^s-(-b)^s$ the period being the least s for which $Q \equiv 0 \pmod{n}$. We make generalizations to the periods of $e-1$ forms in e variables, the case $e=3$ being a modular theory of the two abscissas that can be drawn from a point in the (η, ξ) plane to two surfaces.

19. Mr. E. G. Keller: *A necessary condition for a planet to be of annular origin.*

This paper gives a necessary condition which must be satisfied for a Laplacian ring to have condensed into a satellite. The condition is derived from the moment of momentum and energy equations of the ring, primary, and resulting satellite. The meridian section of the ring is circular or elliptical. The particles of the ring describe circular orbits about the primary according to the Newtonian law. In the first section of the paper the matter of the ring is supposed homogeneous. In the second section the condition is developed for a ring of variable density. The criterion, when applied to the solar system, indicates that Mercury and four satellites each of Jupiter and Saturn can not be of annular origin.

20. Professor G. Y. Rainich: *On the field of radiation.*

The formulas for singular Lorentz transformations given by the author in the February number of the Proceedings of the National Academy of Sciences are applied in this paper to the study of fields which remain invariant under transformations not affecting the path of a light ray. The field of electric and magnetic forces, the field of stresses and the curvature field of the "general relativity theory" are treated from this point of view.

21. Professor P. R. Rider: *The devil's curve and abelian integrals.*

The curve $y^4-x^4+ay^2+bx^2=0$ is called by French geometers *la courbe du diable*. It has been a favorite example in curve tracing. This paper discusses the Abelian integrals connected with the curve. It has appeared in the American Mathematical Monthly (vol. 34 (1927), p. 199).

22. Professor H. T. Davis: *A note on the factoring of Fredholm minors.*

L. Tocchi has proved that a necessary and sufficient condition for the factoring of a Fredholm minor of the nth order is that the minors of lower order shall vanish for some characteristic value of the parameter. The present note derives this theorem from the fact that a necessary and sufficient condition that a function $D(x,y)$ shall be factorable is that it satisfies the equation

$$D(x,y)\frac{\partial^2 D}{\partial x\partial y}-\frac{\partial D}{\partial x}\frac{\partial D}{\partial y}=0.$$

23. Professor Solomon Lefschetz: *On intermediary functions.*

An intermediary function of p variables is a function defined by certain properties of periodicity analogous to those of theta functions. In a paper published in Crelle, vol. 95, Frobenius has made a thoroughgoing investigation of these functions but his methods are exceedingly involved. The main object of this note is to point out that the work of Frobenius can be greatly simplified. Together with a paper by Appell in Journal de Mathématiques, 1891, it may form the basis for a most elegant and direct presentation of the chief theorems on multiply periodic functions.

24. Dr. L. E. Ward: *Expansion of functions.*

The author considers the expansion of functions in series whose terms are solutions of the differential equation $u'''+\rho^3u=0$ and three boundary conditions which are linear and homogeneous in the dependent variable and its first two derivatives with real coefficients. The boundary conditions are restricted so that two (called the first two) bear at one point only and the third at a second point and possibly also at the first point. It is found that a uniformly convergent series of the characteristic functions necessarily converges to an analytic function satisfying two conditions closely related to the first two boundary conditions, and that the region of convergence is the interior of an equilateral triangle centered at the point at which the first two conditions bear. Subject to further minor restrictions on the function expanded, the formal series is proved to converge uniformly to the generating function. Use is made of a contour integral for the sum of the first n terms of the formal series.

25. Dr. L. E. Ward: *Expansion of functions.*

The author considers the problem corresponding to that stated in the preceding abstract but with special boundary conditions of which the first bears at one real point, the second at one complex point, and the third at both points. The characteristic functions are shown to fall into two sets, each set being identical with a set of characteristic functions arising from three boundary conditions of the type considered in the preceding paper. The type of function expansible in uniformly convergent series of these characteristic functions is obtained, and convergence is proved. The corresponding problem with boundary conditions bearing one at each point, two of which are complex, is also considered. The characteristic functions fall into three sets, each set being identical with a set of characteristic functions of the type considered in the preceding paper; and, subject to certain minor restrictions, an arbitrary analytic function can be expanded in a uniformly convergent series of these characteristic functions.

26. Professor Henry Blumberg: *Remarks concerning a theorem on arbitrary real functions.*

At the 1926 Philadelphia meeting of the Society, the author presented a theorem, of which the following is a corollary: If $f(x,y)$ is a real function, l a straight line in the xy-plane, and d_1, d_2 two directions approaching l on the same side, then lim sup $f(x,y)$ as (x,y) approaches a point P of l along d_1 is not less than lim inf $f(x,y)$ as (x,y) approaches P along d_2, with possibly a countable number of exceptional positions of P on l where this inequality may be invalid. If $f(x,y)$ is symmetric in its arguments, in particular, if it is an "interval function," it follows that lim sup $f(x, x+0) \geqq$ lim inf $f(x, x-0)$ except possibly for an enumerable set of x's. By defining various interval functions relative to a given function of a single variable, various theorems are obtained concerning unconditioned functions of one variable, among them the theorem of W. H. Young on the right and left limits of a function (Quarterly Journal of Mathematics, vol. 39), the theorem of G. C. Young on right and left derivatives (Acta Mathematica, vol. 37), the theorems of Blumberg concerning the equality of the right and left saltus (this Bulletin, vol. 24) and the theorem of Kempisty on the approximate limits of a function in the sense of Lebesgue measure (Fundamenta Mathematicae, vol. 6).

27. Professor Henry Blumberg: *On the character of the set of points of a surface where there is a vertical, conical lacuna.*

The "surface" $z=f(x,y)$, where f is any real function whatsoever, is said to have a conical, vertical lacuna at the point $P\equiv(x,y)$, if there is a vertical (double) cone, with vertex P and axis parallel to the z-axis, having no surface points in it other than P in a sufficiently small neighborhood of P. This paper shows that the set of points where there is such a lacuna is an F_σ, i.e., the sum of a countable number of closed sets; and conversely, if S is a point set in the (x,y) plane representable as an F_σ, then a function $f(x,y)$ exists such that $z=f(x, y)$ has a vertical conical lacuna precisely at the points of S but nowhere else.

28. Dr. C. F. Roos: *A general problem in the calculus of variations.*

In order to develop a general theory of depreciation in economics it is necessary to consider a problem similar to a Lagrange problem with variable end points but different from the Lagrange problem in that the variable end values occur in the integrand. The problem somewhat resembles the brachistochrone problem of determining the curve of quickest descent from a fixed curve to a fixed point in a resisting medium. The depreciation problem, the brachistochrone problem just referred to, and a problem proposed by Bolza to include the most general Lagrange problem with general boundary conditions and the Mayer problem with general boundary conditions as special cases, are all special cases of the general problem discussed in this paper. The Euler-Lagrange multiplier rule and the transversality condition for this general problem are obtained by modern calculus of variations methods.

29. Dr. C. F. Roos: *A general problem of minimizing an integral with discontinuous integrand.*

In this paper it is shown that the problem of determining when to replace one machine by another in such a way that there results a maximum profit for a given period of time is a type of Lagrange problem with a discontinuous integrand. The problem differs from the ordinary problem with discontinuous integrand discussed by Bliss and Mason in that the integrand is a function of the end values which are variable. In addition to the Euler-Lagrange multiplier rule and the transversality conditions which follow readily by the analysis of my paper *A general problem in the calculus of variations*, corner conditions and further necessary conditions which lead to sufficient conditions are obtained for a general problem which includes this replacement problem as a special case.

30. Professor J. H. Taylor: *Parallelism and transversality in a subspace of a general (Finsler) space.*

An arbitrary integral of the form $\int F(x, \dot{x})dt$, where F satisfies the conditions for a *regular* problem in the calculus of variations, is taken as the definition of arc length in a space N of n dimensions. In terms of parallelism and angle developed by the author in an earlier paper (*A generalization of Levi-Civita's parallelism and the Frenet formulas*, Transactions of this Society, vol. 27 (1925), pp. 246–264), it is shown that parallelism of vectors in the space N implies parallelism in any subspace of N in which they lie, and that the angle between vectors is the same if measured with respect to the metric of the space N or with respect to the metric of the subspace. Transversality is a special case of orthogonality in the sense here used; hence the transversality condition is preserved in a subspace.

31. Professor J. V. Uspensky: *On the convergence of quadrature formulas related to an infinite interval.*

Denoting by $p(x)$ any positive function, defined in an infinite interval (a, ∞) and such that all the constants $c_n = \int_a^\infty p(x)x^n dx$ exist, the author shows that the quadrature formula of the Gaussian type $\int_a^\infty p(x)f(x)dx = \sum_1^n A_k f(x_k) + R_n$ converges for indefinitely increasing n, whenever the following conditions are fulfilled: 1. There exists a function $\phi(n)$ such that $\phi(n)/n \to 0$ and $c_n(\phi(n)/n)^{2n} \to 0$ as $n \to \infty$; 2. The function $f(x)$ satisfies the condition $|f(x)| < x^m$ for sufficiently large values of x, where m is a positive number, arbitrarily large. The paper states the consequences of this theorem for the convergence of a certain class of continued fractions and for the fundamental theorem in the calculus of probabilities.

32. Professor J. A. Shohat: *A simple method for normalizing Tchebycheff polynomials and evaluating the elements of the associated continued fraction.*

This paper appears in full in the present number of this Bulletin.

33. Dr. N. B. Conkwright: *The summability of Birkhoff series.*

It is known that the sum of the first k terms of a Birkhoff series may be expressed as an integral around a contour C_k which incloses the first k poles of the Green's function associated with a certain linear differential equation and a set of regular boundary conditions at two points a and b. This paper introduces into the integrand functions $\phi(k, \lambda)$, analytic in λ, such that the limit of the integral as k becomes infinite represents an arbitrary function $f(x)$, absolutely integrable in the interval ab. The introduction of this factor is equivalent to multiplying the first k terms of the series by quantities $d_{kl}(l = 1, \cdots, k)$, thus summing the series by a method of mean values with finite reference. If $f(x)$ is continuous in a sub-interval $\alpha\beta$ of ab, the Birkhoff series is uniformly summable by the methods under consideration in the interval $\alpha + \delta \leqq x \leqq \beta - \delta$, δ being a positive constant. The summability of multiple Birkhoff series may be investigated by a repetition of the argument for a single series. The method may also be applied to series arising from boundary conditions applied at more than two points.

34. Professor W. M. Whyburn: *Existence and oscillation theorems for linear non-self-adjoint differential systems of the second order.*

This paper studies the pair of differential equations $y' = K(x, \lambda)z$, $z' = G(x, \lambda)y$ together with boundary conditions that apply to more than two points of the interval. Existence and oscillation theorems are established for cases where one of the conditions involves only one point of the interval while the other involves two or more points. K and G are summable functions of x on $X(a \leqq x \leqq b)$, for each λ on $L(L_1 \leqq \lambda \leqq L_2)$, continuous functions of λ on L for each x on X, and bounded numerically by a function $M(x)$ that is summable on X. Certain non-self-adjoint systems of the two-point type are treated as special cases of these more general systems.

35. Professor W. M. Whyburn: *On the polynomial convergents of power series.*

In a paper published in the Annals of Mathematics, (2), vol. 8 (1906), pp. 189–192, M. B. Porter defined a particular set of convergents for a power series and discussed certain properties of this set. In the present paper it is shown that every point on the boundary of the circle of convergence is a limit point of the zeros of Porter's set of convergents. This work includes the theorem of Jentzsch (see Landau, Funktionentheorie, Berlin, 1916, p. 80) as a corollary. Other properties of these convergents are also discussed.

36. Mr. M. E. Mullings: *First-order vector differential equations of scalar functions.*

This paper demonstrates the existence of a unique solution of the vector differential system $\nabla u(\rho) = \theta(u,\rho)$, and $u(\rho_0) = u_0$ under very general conditions. Components are avoided by use of a set of definitions extending certain notions of real variables to vectors. Four theorems are obtained, the first dealing with continuity. The second shows the existence of the gradient of a scalar function under the given conditions, the third shows the existence of a line integral of the gradient that is independent of the path between two points, and the last theorem secures a unique solution for the given vector system.

37. Professor H. E. Bray: *An auxiliary theorem in the theory of harmonic functions.*

If the function $u(M)$ is harmonic at all interior points M of the unit sphere, then the quantity $F(r,s) = \int u(M)dM$ defines an additive function of segments s, the integration extending over the interior of the segment s, on the sphere of radius $r<1$, and bounded by circles of latitude θ', $\theta''=$ const. and by meridians ϕ', $\phi''=$const. Thus $F(r,s) = F(r,\theta',\theta'',\phi',\phi'')$ is a function of the four variables θ', θ'', ϕ', ϕ'', with parameter r. The theorem is that if $F(r_i,s)$ is of uniformly limited variation for all r_i of a sequence $[r_i]$ converging to unity, then there exist two sets of numbers, $E'(\theta)$ and $E''(\phi)$, dense in the respective intervals $(0, \pi)$ and $(0, 2\pi)$, and a subsequence $[r_i']$ of $[r_i]$, such that lim $F(r_i', \theta', \theta'', \phi', \phi'')$ exists, as $i=\infty$, provided that θ', θ'' belong to $E'(\theta)$ and ϕ', ϕ'' to $E''(\phi)$. The analogous result for the circle was proved by Evans, by the use of Fourier series. This paper shows that the sequence of functions $F(r_i, \theta', \theta'', \phi', \phi'')$ converges in the mean of order 1, as $i=\infty$.

38. Dr. A. H. Copeland: *Note on the Fourier development of continuous functions.*

This paper will appear in full in an early issue of this Bulletin.

39. Professor G. C. Evans: *A general Neumann problem for the sphere.* Preliminary communication.

Let v be harmonic within a sphere S of unit radius, and let w_r be a closed regular curve on the concentric sphere S_r of radius $r<1$, whose projection on S is w. The author considers the problem of determining v so that $\int(\lambda v+\partial v/\partial r)d\sigma$ extended over the region bounded by w_r takes on given values $H(w)$ as r approaches 1, $H(w)$ being a bounded additive function of regular curves on S. For the class of functions for which $\int|\partial v/\partial r|d\sigma$, extended over S_r, remains bounded as r approaches 1, there is a unique solution of the problem, provided λ is not one of a set of characteristic values $\leqq 0$. For the characteristic value $\lambda=0$, there is a unique solution provided $H(S)=0$; in this last case, if $H(w)$ is absolutely continuous the problem reduces to the Neumann problem for boundary values summable (L). The method used is to write v as the potential of a simple distribution of mass, for which the mass function is given in terms of $H(w)$ by means of a Stieltjes integral equation.

40. Mr. J. J. Gergen: *On generalized lacunae.* Preliminary communication.

According to S. Mandelbrojt the generalized lacunae of the series $f(x)=\sum a_n x^n$ are the lacunae of the series $F(x)=\sum g(a_n)x^n$, $g(z)=\sum \alpha_m z^m$ being an entire function without constant term. The author proves theorems about the singularities of a series with generalized lacunae, defining the notion of the exponential order of the singularities of a uniform function. If $f(x)$ has as its only possible singularities the p pth roots of unity and the point at infinity, no singular point of $f(x)$ being of exponential order $>q$, then the only possible singularities of $F(x)$ are those of $f(x)$, provided $|\alpha_m|^{1/m}=\sigma(e^k)$ where $k=-2^{(q+1)m}$. The lacunae of $F(x)$, determining the nature of its singularities, likewise determine the nature of those of $f(x)$. In particular, making use of Carlson's theorem, if the sequence (a_n), with superior limit $|a_n|^{1/n}=1$, is a sequence of complex integral numbers consisting of two infinite subsequences $\{a_{\lambda_n}\}$, $\{a_{\mu_n}\}$ if $g(a_{\mu_n})=0$, if the superior limit of $|g(a_{\lambda_n})|^{1/\lambda_n}=1$, and if $\lim(\lambda_{n+1}-\lambda_n)=0$, then the circle of convergence of $f(x)$ is a cut.

41. Dr. D. V. Widder: *The singularities of a function defined by a Dirichlet series.* Preliminary communication.

The purpose of the present paper is to generalize a familiar theorem of Hadamard concerning the multiplication of singularities of functions defined by Taylor's series. A Dirichlet series, $f(s)=\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, may be regarded as a generalization of Taylor's series. If $\phi(s)=\sum_{n=1}^{\infty} b_n e^{-l_n s}$, then it is found that under certain conditions the function defined by the series $\sum_{n=1}^{\infty} b_n e^{-l_n z}\sum_{\lambda_m<l_n} a_m(l_n-\lambda_m)^{\mu}$ can have no other singularities in the whole

plane but points $\alpha+\beta$ and β, α designating a singularity of $f(s)$ and β a singularity of $\phi(s)$. Here μ is a suitably chosen positive number. The proof is made by replacing the integral of Parseval, of use in the proof of Hadamard's theorem, by the integral $(1/2\pi i)\int_c f(s)\phi(s)/s^{\mu+1}$ suggested to the author by S. Mandelbrojt.

42. Dr. C. C. Camp: *A differential system in p independent parameters which leads to a multiple series development.* Preliminary communication.

For the simplest case $p=2$ one has the system $X'+(\lambda a+\mu b)X=0$, $Y'+(\lambda c+\mu d)Y=0$ and the boundary conditions $X(\pi)=X(-\pi)$, $Y(\pi)=Y(-\pi)$. By assuming $\Delta\equiv A_1D_1-B_1C_1\neq 0$, where $2\pi A_1=\int_{-\pi}^{\pi} a(x)dx$, etc. one insures the existence of a double set of principal parameter values alike for this system and its adjoint. In the biorthogonality condition one has the determinant $a(s)d(t)-b(s)c(t)$ as a multiplier. The transformation $v_1=\lambda A_1+\mu B_1$, $v_2=\lambda C_1+\mu D_1$ leads to a comparatively simple Green's function. In the extension of Birkhoff's contour method the evaluation by integrations by parts is effected by somewhat unusual algebraic rearrangements of the integrands. The system may be arranged so that $\Delta>0$ and if one restricts the coefficients a, b, c, d so that each one is either always of one sign or identically zero a satisfactory development of an arbitrary function $f(x, y)$ of the usual sort is obtained with a double series so arranged that λ, v_1, v_2 become infinite together. The author is extending the theory to include any number of parameters.

43. Mr. H. S. Wall: *On the Padé approximants associated with the continued fraction and series of Stieltjes.* Preliminary communication.

The sequence of odd convergents and that of even convergents of the continued fraction of Stieltjes converge to the same or to distinct functions according as $\sum a_i$, the series of coefficient of the continued fraction, diverges or converges. The writer finds that when $\sum a_i$ converges, every diagonal file of approximants parallel to the principal diagonal in the Padé table for the series of Stieltjes converges, no two to the same meromorphic function. When $\sum a_i$ diverges two cases arise: either every diagonal file converges to the same function, which is analytic except for the whole or a part of the negative real axis, or else in a certain portion of the table they converge to one and the same meromorphic function, and in another portion to different meromorphic functions. The remainder after n terms of a Stieltjes series being a series of the same character, the question arises, whether Stieltjes series exist of which a given Stieltjes series is the remainder after n terms. When $\sum a_i$ converges, such series exist: an infinity for each n; when $\sum a_i$ diverges, and $n=1$, if and only if $\sum a_{2i}$ converges. In this case, when the series exists for a given n, all its coefficients are uniquely determined except the first.

44. Professor Louis Brand: *Vector forms of the Euler equation in the calculus of variations and their relation to the Hilbert integral.*

The integral $\int F(r, T)ds$ over plane curves joining two fixed points is first considered, r being the position vector and T a unit tangent vector in the positive sense. Using normal variations, it is shown that $\nabla_r F - dQ/ds = 0$, where $Q(r, T) = FT + \nabla_T F$, along an extremal; tangential variations yield only an identity in partial differentiation. If ∇_r is always normal to the constant vector e, $e \cdot Q$ is a first integral of the above equation. In a field of extremals for which $T = a$, rot $Q(r, a) = 0$; hence $Q = \nabla W$ and $\int Q \cdot T\, ds$ is independent of the path. This is the Hilbert integral. The curves $W =$ const. are the transversals. If the comparison curves lie on a surface of unit normal n, $\nabla_r F - (n \times dQ/ds) \times n = 0$ along the extremals; $n \cdot \text{rot}\, Q = 0$ in a field and $\int Q \cdot T\, ds$ again represents the Hilbert integral. When $F = 1$ the equation states that the geodesic curvature is zero. For surface integrals $\int F(r, n)d\sigma$ limited by a fixed closed curve, $n \cdot \nabla_r F + \text{Div}(Fn + \Delta_n F) = 0$ over the extremals; here Div denotes the *surface* divergence. In a field for which $n = c$, div $P = 0$, where $P = Fc + \nabla_0 F$ and the invariant integral is $\int P \cdot n\, d\sigma$. When $F = 1$ the equation states that the mean curvature is zero.

45. Professor J. B. Shaw: *On linear algebra.*

The main developments of linear algebra have arisen from the consideration of the hypernumbers as linear operators, whence the characteristic equation and resulting properties. The present investigation is based upon the properties of idempotents and nilpotents, that is, the hypernumber is viewed from the standpoint of quadratic forms; a study of the structures of algebras (non-associative and associative) shows the existence of an ever increasing complexity.

46. Professor I. A. Barnett: *Conformal transformations in function space.*

G. Kowalewski has found the most general regular infinitesimal transformation possessing the property of preserving the angle between two curves in function space. In this paper, the class of all finite transformations which can be generated by a certain subgroup of the above transformations is obtained. Analytically, this amounts to solving a set of linear integro-differential equations. These equations are completely solved and it is shown that the finite transformations generated by the above infinitesimal conformal transformation constitute a one-parameter family of non-singular conformal transformations.

47. Professor Dunham Jackson: *A problem in minima.*

Various problems have arisen by generalization of the least-square property of Fourier series, according to which the partial sum of the series, regarded as an approximation to the given function $f(x)$, is distinguished among all trigonometric sums of like order by the fact that the integral of the square of the error is a minimum. If the integral is looked upon as a rudimentary Gramian determinant, of the first order, one form of generalization is obtained by supposing that two functions $f(x)$ and $\phi(x)$ are given, and seeking a trigonometric sum $T_n(x)$, of specified order n, to minimize the Gramian of the functions $f(x)-T_n(x)$ and $\phi(x)-T_n(x)$. In the geometry of function space, the elementary problem calls for the minimizing of a distance, and the new one for the minimizing of the area of a triangle. This paper discusses the existence of a solution of the triangle problem for specified n, and the convergence of the sums $T_n(x)$ which it defines, as $n\rightarrow\infty$.

48. Dr. Lulu Hofmann: *On some special congruences of rays connected with analytic functions.*

Given an analytic function $w=f(z)$, we choose two arbitrary planes (w) and (z) forming an angle α in ordinary three-dimensional space for the Gaussian planes of w and z and study the congruences of rays obtained by joining two corresponding points of (w) and (z). (For the general investigation of such congruences in the case of parallel planes, see Wilczynski, Transactions of this Society, 1919.) For a general analytic function invariance with regard to α is stated for the curves cut out in (w) and (z) by any developable of the congruence and for the anharmonic ratio formed on any line of the congruence by its focal points and its intersections with (w) and (z). The congruences of the functions $w=cz^k$, where k is a positive integer, are examined in detail for the special position of the two Gaussian coordinate-systems in which the two points of origin and the two real axes coincide. Under the same assumption the congruence of $w=c/z$ is studied. The focal surfaces are investigated by means of the representative system of plane curves according to the methods of algebraic geometry.

49. Professor H. A. Simmons: *On the existence of a solution of the Diophantine equation* $\sum 1/(x_1\cdots x_s)=1$.

The papers by Kellogg (American Mathematical Monthly, 1921, p.300–303) and Curtiss (ibid., 1922, p. 330–337) on the Diophantine equation in n variables $\sum 1/x_1=1$, led the author to study the equations $\sum 1/(x_1x_2)=1$, $\sum 1/(x_1x_2x_3)=1$. In obtaining these solutions certain principles appeared which are independent of the number of variables in each unit fraction. Thus he was led to a solution of the general equation $\sum 1/(x_1x_2\cdots x_s)=1$, s any positive integer $\leqq n$. The solution obtained for this equation is a perfect analog of Kellogg's solution and indeed reduces to it in the case $s=1$. On account of Curtiss's proof that Kellogg's solution contains the maximum x that can appear in a solution of the equation $\sum 1/x_1=1$ and additional facts, the author believes that his solution contains the largest x that can appear in a solution of the equation $\sum 1/(x_1x_2\cdots x_s)=1$.

50. Professor J. S. Turner: *Positive determinants of the Seelhoff type.*

The present paper contains a number of positive determinants which can be used in a similar manner to those of Seelhoff for testing the prime of composite character of large integers.

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