MATHEMATICS AND THE BIOLOGICAL SCIENCES*

BY H. B. WILLIAMS

We meet this afternoon to do honor to the memory of one whose influence on the development of scientific thought and achievement has been in many ways remarkable. The auspices under which this assembly has gathered is in itself an earnest of the regard in which the name of Josiah Willard Gibbs is held by the mathematicians of America. Forty years ago he delivered the vice-presidential address before the Section of Mathematics and Astronomy of the American Association for the Advancement of Science. The topic was *Multiple Algebra.*† His own contributions to this subject, particularly his development of the theory of dyadics, would be sufficient to establish without question his standing as a pure mathematician, while his vector analysis with its practical and convenient notation has been of no small service to the cultivators of mixed mathematics.

The soundness of his judgment in the field of physics is attested by the fact that he was among the first to take up and extend Maxwell's electromagnetic theory of light. In his obituary of Professor Gibbs it is remarked by Professor Bumstead‡ that these optical papers are noteworthy for the entire absence of special hypotheses regarding the connection between matter and ether. It seems to have been a characteristic trait with him to strip his problems of every unnecessary element before commencing extensive treatment of them.

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* The Fourth Josiah Willard Gibbs Lecture, read at Philadelphia, December 28, 1926, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

† J. Willard Gibbs, *On multiple algebra,* Proceedings A. A. A. S., vol. 35 (1886), pp. 37–66. Also Gibbs, Collected Scientific Papers, vol. 2, p. 91.

‡ Bumstead, Henry A., *Josiah Willard Gibbs,* American Journal of Science, (4), vol. 16 (Sept. 1903). Also Gibbs, Scientific Papers, vol. 1.

Professor Hastings relates that Gibbs once replied to a complimentary remark about his work that if he had met with any success it was largely because he had found ways to avoid the mathematical difficulties! I am inclined to think that this was not intended altogether as a pleasantry. For although Gibbs met and overcame many serious mathematical difficulties in his work, he probably was well aware that he had avoided much unnecessary labor by this habit of first eliminating all but the bare essentials of his problem. Professor Bumstead remarks (loc. cit.) of these optical papers that it seems likely the considerations advanced in them would have sufficed firmly to establish the electromagnetic theory even if the experimental discoveries of Hertz had not supplied a more direct proof of its validity.

It is, however, his work in thermodynamics, particularly the great work, *On the equilibrium of heterogeneous substances*,* which has made the name of Josiah Willard Gibbs familiar in every part of the world where science is cultivated. The present year marks the fiftieth anniversary of the publication of the first part of this paper, the importance of which was not at once recognized. We shall hardly be surprised at this when we remember that the subjects treated are chiefly important to the chemist. Indeed this work may justly be said to have laid the foundation for theoretical chemistry. But in 1876, and for a good many years thereafter the number of chemists who had more than a nodding acquaintance with even the more elementary branches of mathematics was comparatively small. The paper is not only highly mathematical, but the methods employed are such as to make its reading a privilege open only to those who have had the foresight and industry to acquire a considerable mathematical armamentarium.

It is usually stated that the paper remained in comparative oblivion until its value was discovered by Professor Ostwald,

* Transactions of the Connecticut Academy of Arts and Sciences, vol. 3, pp. 108–248 and pp. 343–524. Also Gibbs, Scientific Papers, vol. 1, No. 3, pp. 56–353.

who, in 1891, translated it into German. However, I find in the presidential address of the late Lord Rayleigh,* given before the British Association at Montreal in 1884, the following comment, made in connection with his discussion of the dissipation of energy: "The foundations laid by Thomson now bear an edifice of no mean proportions, thanks to the labors of several physicists, among whom must be especially mentioned Willard Gibbs and Helmholtz. The former has elaborated a theory of the equilibrium of heterogeneous substances wide in its principles, and we cannot doubt, far reaching in its consequences." It thus appears that Lord Rayleigh had read and was fully cognizant of the importance of this paper prior to the Montreal meeting of 1884. His comment is at once a tribute to the genius of Willard Gibbs and a proof of the catholicity of his own scientific interests. In 1899 this paper was translated into French by LeChatelier and in 1906 it became more widely available to English readers through the publication of Professor Gibbs' collected Scientific Papers. To the influence of this paper it is largely due that chemistry, then an almost un-mathematical science, has so developed that the mathematical equipment now required by the student of chemistry differs but little from that which is requisite for the student of physics. The two subjects tend ever more and more to overlap.

Nor is it alone in theoretical chemistry that the influence of Gibbs is felt. It is hardly too much to say that chemical engineering and the whole vast edifice of chemical industry rest upon his work as their secure foundation.

The esteem in which Gibbs was held is reflected in many comments which may be found in the scientific literature. It is somewhat unusual to apply adjectives of commendatory character when quoting the work of another investigator and when this *is* done by a recognized master in his field, it signifies more than ordinary appreciation of the validity and

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* Lord Rayleigh, Presidential Address, British Association Report, Montreal, 1884, pp. 1–23. Also Rayleigh, Scientific Papers, vol. 2, p. 342.

importance of the work. Quoting again from the works of Lord Rayleigh, in his paper, *On reflexion of light at a twin plane of a crystal*,* we find mention of "Professor Gibbs' excellent comparison of the elastic and electric theories of light with respect to the law of double refraction and dispersion of colors." Again in the paper on *Foam*† Lord Rayleigh alludes to "the masterly discussion of liquid films by Professor Willard Gibbs." In the review of Gibbs' collected papers by Jeans is the following: "The publication of the collected works of a great scientist serves the double purpose of forming a memorial to the genius of the writer and of increasing the usefulness of his labors by making his writings available to other workers. It is in connection with the second of these purposes rather than the first that one is inclined to consider the present volumes. They form, it is true, a memorial, and a fitting memorial to the greatness of Professor Gibbs' scientific work, but we feel that a memorial was hardly needed—his greatness is too well established and his work too well known for either to be enhanced at this stage by the publication of volumes of paper and ink."

In view of the fact that Professor Gibbs has exerted his greatest influence through the application of mathematical methods to the problems of a branch of science which, prior to his work, had not been regarded as a fertile field for theoretical cultivation, it seems not inappropriate that in meeting here to renew our recognition of the fundamental importance of his work we should turn our attention to the possibility of extending profitably the application of mathematical modes of thought to yet another field of scientific endeavor. To many the idea of applying mathematical methods to biological investigation may seem quite as unusual as did the idea of their application to chemistry at the

* Philosophical Magazine, vol. 26, p. 241; and Scientific Papers, vol. 3, p. 190.

† Proceedings of the Royal Institution, vol. 13 (1890), pp. 85–87. Also Scientific Papers, vol. 3, p. 359.

beginning of the last quarter of the nineteenth century. A tacit implication that the biological sciences are essentially un-mathematical subjects is conveyed by the titles under which the two sections of the Proceedings of the Royal Society of London are published. Section A is supposed to contain the mathematical and physical papers, while section B is the biological section. Notwithstanding this implication and the rather widespread impression that the problems of biology are too complicated or too indefinite, or both, to permit of mathematical treatment, certain of its questions have been so investigated and such studies still continue. Among those who have been successful in the application of mathematics to physiology may be mentioned no less a personage than Helmholtz. The mathematicians in my hearing all know of Helmholtz as a mathematician and the physicists know that he was regarded as one of the ablest physicists of his day, but because of his eminence in these fields it is easy to forget that he began his career as a physician and that his first scientific post was the professorship of physiology at Koenigsberg. From Koenigsberg he went to Bonn as professor of anatomy and physiology and from Bonn he was called to the professorship of physiology at Heidelberg. It was during these years that the ophthalmoscope was discovered and the great works on *The Science of Tone Perception* and on *Physiological Optics* were written. The latter work has recently been translated into English fifty years after its original publication. Although the translation was undertaken partly in commemoration of the centenary of his birth, it was welcomed more because this book is still, in many aspects of the subjects treated, the most authoritative work available. That it should remain a living book during all this period of intense scientific activity is probably due in no small measure to the precise habits of thought of the great investigator who wrote it. In this and in the work on sound will be found a wealth of examples of the application of mathematics to physiological problems. One might name a goodly number of men among the older school of physiologists who made use of mathematics in various aspects of their work.

Shortly before the middle of the nineteenth century there began an epoch of great activity and productivity in organic chemistry. It was not long before the trend of thought and investigation in physiology began to show the influence of this activity in chemistry and physiological investigations have ever since tended to proceed more and more along chemical lines. As chemistry had not yet evolved into a mathematical science, it was not unnatural that physiologists should have tended away from mathematics in their training and there was a considerable period when this was largely true. There have always been a few individuals who either by chance or from personal predilection have brought to the study of physiology a considerable mathematical training and an occasional one who did so from conviction that this training is important for the most successful pursuit of his profession. In more recent years, and particularly since the advent of theoretical chemistry, there has been a growing tendency to reinstate mathematics as an essential part of the physiologist's intellectual equipment.

Looking now to the work of contemporaneous writers, the name of Gullstrand is pre-eminent in the field of physiological optics. One can hardly be said to be fully conversant with the present state of thought in this subject unless he has read the more important of Gullstrand's papers, * yet there are few ophthalmologists and not many physiologists who are able to read them and for the same reason that the chemists of Gibbs' day were unable to read his work. Gullstrand's

* Allvar Gullstrand, *Die reelle optische Abbildung*, Kungl. Svenska Vetenskapsakademiens Handlingar, vol. 41 (1906) No. 3; *Die optische Abbildung in heterogenen Medien und die Dioptrik der Kristal-linse des Menschen*, ibidem, vol. 43 (1908), No. 2; *Über Astigmatismus, Koma und Aberration*, Annalen der Physik, (4), vol. 18 (1905), pp. 941–973; *Tatsachen und Fiktionen in der Lehre von der optischen Abbildung*, Archiv für Optik, vol.1 (1907), p. 2; *Preparation of non spherical surfaces for optical instruments*, Kgl. Svenska Vetenskapsakademiens Handlingar, vol. 60 (1919), p. 155, abstracted in Zeitschrift für Instrumentenkunde, vol. 41 (1921), pp. 123–25.

investigations have already led to important practical results. I need only mention his ophthalmoscope, his special lenses for patients who have undergone operation for cataract and the slit-lamp microscopy of the living eye. The influence of his work would undoubtedly be more widespread were there more among the group of men interested who could read his papers. In one of the appendices to the recent translation of Helmholtz's *Physiological Optics*, Gullstrand has presented some of his conclusions freed to a great extent from the more difficult mathematics, but it is unfortunately not possible to present such a subject in the most convincing manner without the logical processes whereby the theory has been built up.

In most of our colleges and universities we require of science students, particularly those who are entering the graduate and professional schools, a reading knowledge of German and French. This is an admirable requirement and the reason for it is too well understood to need explanation. It is less well understood that not alone the chemist and physicist, but the biologist as well, must be able to read mathematical papers if he is not to be cut off from the possibility of understanding important communications in his own field of science. And the situation here is worse than it is in the case of inability to read a foreign language. For a paper in a foreign language may be translated, but in many cases it is impossible to express in ordinary language symbols the content of a mathematical paper in such a way as to convey a knowledge of the logical process by which the conclusions have been reached. The result may be accepted on faith, but this is repugnant to the scientifically trained mind.

Among recent outstanding contributions to physiology is the work of Professor A. V. Hill on the phenomena of muscle contraction which was crowned with the Nobel prize.* One might have thought it unlikely that new information of

* A. V. Hill and W. Hartree, *The four phases of heat-production of muscle*, Journal of Physiology, vol. 54 (1920), pp. 84–128.

fundamental importance would be unearthed in a field on which so much labor had already been bestowed. Our interest lies not so much in the ingenious methods devised by Hill as in the fact that they could have been devised and employed for the successful accomplishment of the purpose only by one who brought to the task the viewpoint and power of the skilled mathematician, trained in the concepts and methods of physics and physical chemistry. The absence of any formidable array of mathematical symbols from Professor Hill's papers might lead the casual reader to surmise that they are of the usual descriptive biological type. A more careful perusal will show that he has endeavored to present his results so far as possible without the use of mathematical symbols and phraseology in order that the papers may be intelligible to the physiologists of the present day. To do this he has been obliged to omit certain details, the nature of which can be readily surmised by one with some mathematical experience.* Professor Hill's training in mathematics and physics followed the well known traditions of Cambridge. That with this preparation, so unusual at the present time for a biologist, he should, from the very beginning of his productive work, have secured results of far reaching importance, carries a lesson which deserves to be well pondered by those whose duty it is to advise young men preparing for a lifework in biological research.

In any enumeration of the applications of mathematics to biological investigation the statistical treatment of biological data should occupy a prominent position. This field is at once one of the most important and one of the most treacherous into which the biological mathematician can

* Since the above was written, it has been possible to verify the accuracy of this surmise. Furthermore, though no mention of it is made in Dr. Hill's paper, the solution of an integral equation was part of the underlying mathematical structure essential to the success of the investigation. Although the inclusion of these details might have deterred many contemporaneous biologists from attempting to read the paper, it seems proper that the facts should be known because of the influence which they may, and should, exert on the rising generation.

venture. Gibbs frequently quoted in this connection a warning of Maxwell and I can do no better than repeat a passage taken from his obituary of Professor H. A. Newton, the astronomer. This was read by Gibbs before the National Academy of Sciences in 1897.* He says "This kind of investigation Maxwell has called statistical, and has in more than one passage signalized its difficulties. The writer recollects a passage of Maxwell which was pointed out to him by Professor Newton, in which the author says, that serious errors have been made in such inquiries by men whose competency in other branches of mathematics was unquestioned." Gibbs was himself a master of this type of investigation and the biological sciences are fortunate in having secured the services of a former pupil and later distinguished colleague of Gibbs, Professor E. B. Wilson, who is now connected with the Harvard Medical School. Time will not permit a lengthy enumeration of the problems which have been approached by this method. Those who are interested will find ample references in the works of Professor Karl Pearson and in those of Professor Raymond Pearl. A recent interesting achievement in the field of statistical biology is the development by Mr. Arne Fisher† of a method of constructing from biological data a series of frequency curves for the death rates at various ages from certain groups of causes. These he has been able to combine into a compound frequency curve from which may be predicted the expected deaths at various ages in an entire population without knowledge of the total number of lives exposed to risk. The resulting curves have been shown to be in good agreement with curves established by the usual actuarial methods for the same populations. The method has been

* J. Willard Gibbs, *Hubert Anson Newton*, American Journal of Science, (4), vol. 3 (1897), pp. 359–376. Also Gibbs, Scientific Papers, vol. 2, pp. 268–284.

† Arne Fisher, *Frequency Curves*, New York, The Macmillan Co., 1922; *Note on a new method of construction of mortality tables when the number of lives exposed to risk is unknown*, Skandinavisk Aktuarietidskrift, 1925 pp. 163–215.

subjected to vigorous attack by actuaries unfamiliar with the biological principles involved, but there can be little doubt of the essential soundness of the procedure and its applicability is by no means limited to actuarial studies.

The use of statistical methods in biological studies has hitherto been confined largely to such matters as are included in the term biometry. Among the various items would be such as birth and death rates, incidence of morbidity, curves of population growth, correlations between various classes of biometric data and the like. These subjects are of great importance, an importance which I recognize fully at the same time that I feel impelled to offer the suggestion that statistical mathematics will perform its greatest service for the biological sciences in a very different field, a field as yet practically untouched. It may even be premature to suggest this application in the present state of our quantitative biological data. In a mathematical study of any system in which it is possible to measure the total integrated changes of energy or of material state, but impossible, on account of its complexity, the smallness of its parts or their inaccessibility to our methods of investigation, to know in detail all of the elements which contribute to the total change, we must have recourse to the statistical method. The brilliant success with which Maxwell has applied this method to the kinetic theory of gases and its power in the hands of Gibbs in investigating the energetics of chemical reactions, entices us toward the thought—the hope, perhaps—that one day it may come to be applied to such studies as the energetics of secretion and other biological processes. If the whole complicated mammalian organism operates in accordance with the principle of the conservation of energy, as experiments with the respiration calorimeters show it to do, it seems altogether likely that the minute details of the energy exchanges will be found to obey the same laws that hold in the world of the non-living. Only in the living organismone must reckon with the effect of the structure. This seems to be the chief peculiarity of living things and in any attempt to apply the methods of chemistry and physics to the problems of biology one must keep constantly in sight the fact that structure and its consequences is an integral part of the problem. It is an item which seems to increase the difficulties enormously. Much data have already been accumulated. It is possible that some further information regarding *structure* is needed before the next step can be taken, possibly something about finer structure than we have been able to study with the microscope hitherto.

My colleague, Professor E. L. Scott, has recently carried out an investigation on the *number* of experiments which must be performed before the results acquire real significance. Such a study is partly mathematical, but there must enter into it experimental data, since the answer is intimately bound up with the uniformity with which the experimental conditions can be controlled and repeated and the certainty with which the experimenter can know that he has controlled the conditions within any specified limits of uniformity. In three series of observations, each in a different biological class, Dr. Scott finds that the number of experiments required for reasonable certainty is of the order of fifty. It is too early to predict whether this order of magnitude applies to any considerable number of types of biological experiment. However, it raises at once the question whether in any short series of biological observations the variations which occur can be regarded as due necessarily to experimental procedure. The variability which Professor Scott has found in the three types studied would seem to make it incumbent upon any investigator who desires to draw important conclusions from short series of observations to show that in the type of work he is doing, the variability is small enough to justify his procedure. In order to do this he must run at least one much longer series of experiments, since at times the larger variations may fail to occur in the first five or ten observations. A small dispersion in a series of five experiments may be sadly misleading.

If we are to begin to make biology an exact science, one of the first requisites will be reliable data. Familiarity with the theory of errors and a critical attitude in drawing conclusions from measurements are characteristics of the men who cultivate those sciences we now regard as "exact." We shall have taken an important step when biologists regard their results in the same critical light. Teachers of mathematics can probably further this desirable situation by explaining to students, who are known to contemplate a career in one of the biological sciences, the importance of a knowledge of the theory of measurements.

From time to time physiologists are constrained to enter upon mathematical investigations not so directly related to biological processes themselves. It sometimes happens that correct interpretation of experimental results requires a study of the theory of the instruments used in making the measurements. At times these inquiries have resulted in substantial improvements in methods of work and design of instruments. Notable examples are to be found in the work of Professor Einthoven in the field of electrophysiology and in that of Professor Otto Frank in hemodynamics.

Physiology is sometimes defined as the physics and chemistry of living things. While this definition resembles most definitions in that it is somewhat lacking in completeness, it serves to direct attention to the particular phases of physiology in which it appears most probable the greatest progress of the immediate future will be made. This is not my personal view alone. A distinguished zoologist recently voiced quite similar opinions in my hearing. That the plant physiologists have reached similar conclusions is evidenced by the attitude of Hampton and Gordon,* who in a recent issue of Science, plead for better preparation in mathematics, physics and chemistry as a foundation for research in plant physiology. They quote Lepeschkin as holding the view that plant physiology must develop hand in hand with

* H. C. Hampton and S. C. Gordon, *A suggested course in plant physiology*, Science, vol. 64 (1926), pp. 417–419.

physics and chemistry. I can see no escape from the conclusion that the physiologist of the immediate future must be to a considerable extent a chemist and physicist and these he can be only by building from a suitable mathematical foundation. Nor is this a hopeless prospect. Textbooks of physics of seventy-five years ago were much larger than at present. This in spite of the enormous additions since made to our knowledge of the subject. But these older books were voluminous because of minute descriptions of phenomena which we now recognize as what a mathematician would call particular cases, comprehended under broad general principles. Chemistry has also grown greatly in detail, but study of the phenomena of radioactivity and the discovery of the isotopes have already done much to simplify its concepts and we may look forward confidently to a time when our successors will be able to learn the broad principles of chemistry much more readily than can be done at present.

The ultimate object of all scientific study is to enable us to predict that in certain circumstances certain events will happen. The old saying, "History repeats itself," is tantamount to recognition that even so complicated a system as large groups of human beings reacts similarly in similar circumstances. We are not yet able to predict in biological matters to the same order of precision as in physics and chemistry, but that is no proof of lack of precision in biological reactions. Rather it points to our inability to control conditions and to recognize small, but significant, points of difference. Perhaps we may look forward to a day when the history of generalization in chemistry and physics may repeat itself in the biological sciences.

Cognate with the problem of securing a more general appreciation of the value of mathematical methods in biological investigation is that of overcoming a certain feeling of aloofness from these methods which in many people amounts almost to a pathological phobia. That the study of mathematics presents difficulties no one will deny, but that the difficulties are so great as many imagine them to be is certainly *not* true of those branches most applicable to the problems of the physical world as we see them today.

There seems to be a rather widespread opinion that some individuals are endowed with an innate capacity for mathematical thinking which is quite as definitely lacking in others. There is probably *some* basis for this opinion. If we think of natural adaptations of more evident type we can find plenty of analogies. For example, no amount of training would enable a heavy Clydesdale horse to compete successfully against a thoroughbred in a running race. Nevertheless the Clydesdale *can* run and given a little more time would cover the same ground as the animal better adapted for speed. I fear that many students, quickly fatigued by the sustained mental effort requisite to follow the reasoning in a new branch of mathematics, conclude too readily that they are of the group who have not an hereditary capacity for this discipline. There are probably not a few professional mathematicians whose inherited capacity is actually less than that of men who have quit the subject in despair. The man of high courage and determination often succeeds in spite of heavy handicaps, while in every field of human endeavor we see failure, often after an auspicious beginning, where these important qualities are lacking. Where courage and industrious habit is combined with large natural capacity we may expect an unusual measure of success, but I am inclined to think that there are few men—or women—whose intellectual capacity is sufficient to permit their successful cultivation of any branch of science, who could not, if they felt an urgent need for it, obtain a sufficient mastery of those branches of mathematics most requisite for the study of the natural sciences.

Before any person of sound judgment will undertake a difficult task he must first be convinced that it is worth doing and that he has a reasonable chance of success if he undertakes it. I have known more than one man of mature years with very meager training in elementary mathematics who, finding himself checked by this lack, has acquired the desired facility by study, aided only by books, and usually crowded into the routine of a busy life. The rapidity with which such a student often progresses depends, it seems to me, very greatly upon the maturity of his mental processes. He knows exactly why he wants the information; he sees beyond the task of the moment and has from the beginning a rather clear conception whither his study is to lead him. Furthermore he is likely to have attained to a certain measure of self-discipline and is able to keep his thoughts from wandering. This requires no effort at all once a real interest in the subject has been awakened. I have known of one instance where such auto-instruction has been attended with marked success in a man who as a youth regarded himself and was regarded by his teachers as one of those who have no capacity for advanced mathematical study. When it becomes necessary to spend the time of maturer years in thus acquiring what might have been secured in youth, the process always involves a certain waste. The individual has been deprived for a number of years of an intellectual aid which he might have had at his service, and the time of his maturity might be better employed in occupations more suited to his riper experience. It seems that the lesson to be derived is that the awakening of a very real interest on the part of the student in the subject of his study is quite as necessary for his success as is the possession of innate capacity.

One aspect of mathematical study which it seems does much to discourage the beginner is the smooth and finished character of every demonstration he meets. Realizing keenly that he could hardly hope to carry through any such perfectly finished logical reasoning on an original problem, he becomes discouraged. He thinks that these textbook demonstrations are models of the way in which mathematicians do their work. Even the artifices employed awaken mingled feelings of admiration for the mind that can conceive and apply them with such wonderful facility, and despair that he will ever be able to match this performance when confronted with an original problem with no one at hand to suggest the manner of approach. The wise teacher can do a great deal at this stage for his encouragement. He may be told that even a finished mathematician usually writes down on his work-paper much more than he subsequently permits to be printed. Also that not infrequently when a result has been reached by devious ways and after expenditure of much time and labor, it is afterward seen to be capable of direct approach. He will thus become aware that the smooth and concise demonstration is not necessarily conceived in that finished form. I wonder if, for its educational value, it might not be better at times to let students see how some of these results which can be expressed so compactly and neatly when we know just how to do it, were actually reached originally. Beginners can hardly be expected to go to the primary sources, but the teacher who is familiar with the original literature can often perform a real service to his students by bringing to their attention passages from the older authors who, writing at a time when knowledge of their subject had not yet been cast into its present condensed form, sometimes shed light on points which give difficulty in the modern standard treatises.

I am reminded of a passage in Maxwell's *Electricity and Magnetism.** Referring to the brilliant demonstration by Ampère of the laws of action between electric currents, he says;

"The experimental investigation by which Ampère established the laws of the mechanical action between electric currents is one of the most brilliant achievements in science. The whole, theory and experiment, seems as if it had leaped full grown and full armed from the brain of the 'Newton of electricity'. It is perfect in form, unassailable in accuracy and it is summed up in a formula from which all the phenomena may be deduced, and which will always remain the cardinal formula of electrodynamics. The method of Ampère, however, though cast into an inductive form, does not allow us to

* J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Oxford, Clarendon Press, 1873, vol. 2, p. 162, § 528.

trace the formation of the ideas which guided it. We can scarcely believe that Ampère discovered the law of action by means of the experiment which he describes. We are led to suspect, what indeed, he tells us himself, that he discovered the law by some process which he has not shewn us, and that when he had afterwards built up a perfect demonstration, he removed all traces of the scaffolding by which he raised it. Faraday on the other hand, shews us his unsuccessful as well as his successful experiments, and his crude ideas as well as his developed ones, and the reader, however inferior to him in inductive power, feels sympathy even more than admiration, and is tempted to believe that, if he had the opportunity, he too would be a discoverer. Every student therefore should read Ampère's research as a splendid example of scientific style in the statement of a discovery, but he should also study Faraday for the cultivation of a scientific spirit, by means of the action and reaction which will take place between newly discovered facts and nascent ideas in his own mind." Although written by a physicist about a question in physics, it seems to me similar considerations apply to mathematics. If the student can be made to realize at the outset of his work that the beautiful edifice he admires has not arisen by some species of black magic out of nothing, but was originally cumbered by the scaffolding and debris of construction, he will learn that he must expect to accumulate a certain amount of debris when he undertakes to produce something of an original character.

A difficulty which not infrequently stops the novice in reading mathematical papers lies in his attempting to read from formula to formula, thinking that because the writer has written them in immediate succession, he should be able to see the derivation at once without the necessity of performing any work with paper and pencil. A reader of long experience with the subject under discussion may be able at times to do this, but the student should be cautioned in every case where it is not "evident" to start manipulating with paper and pencil the last formula he understands until the next succeeding one has been derived. In writing papers which will probably be read only by professional mathematicians, authors not infrequently omit so many intermediate steps in order to condense their papers that the filling in of the gaps even by industrious use of paper and pencil may become no inconsiderable labor, especially to one approaching the subject for the first time. In reviewing the Electrical Papers of Oliver Heaviside, Professor FitzGerald remarks:* "In his most deliberate attempts to be elementary, he jumps deep double fences and introduces short cut expressions that are woeful stumbling-blocks to the slow-paced mind of the average man. . . ." Those who write for biological readers of the present time should bear in mind that the saving of effort which the insertion of a few, possibly not altogether necessary, formulas will effect may make all the difference between success in comprehending the paper and utter discouragement. Since papers are printed to be read and to get results, the saving of a little paper and ink at the expense of great loss of clarity is surely no gain.

It may not be wholly uninteresting to the mathematicians and possibly may shed light on the value of mathematics for those less familiar with its possibilities to advert briefly to some considerations of the subject colored by a physiological background. It is well known that man is set apart from other animals by reason of his greater intellectual powers. Wherein is the source of these powers? There are structural differences between the brains of the most highly developed of the lower animals and that of man, but when one considers the great disparity of function he is struck by the relative smallness of the structural difference. There is evidence, which I cannot stop to present, that the memory of the lower animals is to a great extent, if not entirely, a memory of particular concrete instances. Through the development of language man has substituted for these, abstract and general symbols which represent any object or event of a

* G. F. FitzGerald, *Heaviside's electrical papers*, Electrician, August 11, 1893. Also G. F. FitzGerald, Collected Scientific Writings, No. 61, p. 293.

given class. The phenomena of aphasia give evidence that certain limited parts of the brain are concerned with the language function and that thinking is ordinarily done in language symbols, usually in sound symbols. It is evident how much more easily the processes of thought can be extended in general symbols than would be the case if all thought had to deal with memories of particular and concrete objects. Now mathematics is both a body of truth and a special language, a language more carefully defined and more highly abstracted than our ordinary medium of thought and expression. Also it differs from ordinary languages in this important particular: it is subject to rules of manipulation. Once a statement is cast into mathematical form it may be manipulated in accordance with these rules and every configuration of the symbols will represent facts in harmony with and dependent on those contained in the original statement. Now this comes very close to what we conceive the action of the brain structures to be in performing intellectual acts with the symbols of ordinary language. In a sense, therefore, the mathematician has been able to perfect a device through which a part of the labor of logical thought is carried on outside the central nervous system with only that supervision which is requisite to manipulate the symbols in accordance with the rules. It is not even necessary during the intermediate mathematical steps to inquire what meaning is to be attached to the groupings of symbols. The thought which I have here in mind has been well expressed by Maxwell,* in referring to the work of Lagrange. Maxwell says: "The aim of Lagrange was to bring dynamics under the power of the calculus. He began by expressing the elementary dynamical relations in terms of the corresponding relations of pure algebraical quantities, and from the equations thus obtained he deduced his final equations by a purely algebraical process. Certain quantities (expressing the reactions between the parts of the system called into play by

* J. C. Maxwell, *Electricity and Magnetism*, Ed. of 1873, vol. 2, p. 184, § 554.

its physical connexions) appear in the equations of motion of the component parts of the system, and Lagrange's investigation, as seen from a mathematical point of view, is a method of eliminating these quantities from the final equations. In following the steps of this elimination the mind is exercised in calculation, and should therefore be kept free from the intrusion of dynamical ideas. Our aim, on the other hand, is to cultivate our dynamical ideas. We therefore avail ourselves of the labours of the mathematicians, and retranslate their results from the language of the calculus into the language of dynamics, so that our words may call up the mental image, not of some algebraical process, but of some property of moving bodies." These remarks of Maxwell would apply just as truly had the particular dynamical phenomena he had in mind been those of protoplasm.

One often hears it remarked that no new knowledge can be obtained by mathematical processes. The same may be as truly said of other modes of thought in the only sense in which it is true of mathematics. Thought cannot begin until there is material to think about, but the same material is presented alike to the animal, the savage, the child, and the scholar. No one who has given the matter serious thought will deny that the intellectual processes, whether mathematical or otherwise, do discover important truth in handling the material presented by the external world. It is the important *relationships* between the facts of observation which are discovered by thought and form the basis of our intellectual lives. Of all the aids to thought that man has devised the most powerful is mathematics. So long as thinking must be carried on solely in memories of particular things or of their corresponding sound symbols it is difficult to be certain that one has made a reasonably complete survey of the relationships between the facts, that he has realized all of the implications contained in them. Formulation of the results in a mathematical expression may not quite adequately represent them. Usually it represents a system simplified in order that the corresponding mathematical expressions may not be beyond our powers of manipulation. If the behavior of the simplified system can be regarded as sufficiently similar to that of the real one which we wish to study, we have secured a powerful aid in seeking the implications of the facts which are at our disposal. Whenever any of the natural sciences has been able so to formulate its observed facts, progress in that branch has been accelerated in a manner so remarkable as to leave no room for doubt of the efficacy of the mathematical method. The greatest need of the biological sciences today is to discover amongst the wealth of empirical facts those relationships which are most significant; those whose recognition will result in the transfer of long lists of particular items to a few general classes, thus simplifying our views regarding them.

The physiologist deals with those aspects of life which are subject to observation and measurement by the same instruments and methods that are used in the study of non-living matter. If there be vital phenomena of a transcendent character, these do not concern the physiologist as a physiologist. So soon as the physical and chemical data, including the structural relationships, have been established with reasonable certainty, the work of the mathematician may begin. There is still much uncertainty about some of the data and their improvement by critical study is perhaps the first requisite. In this we shall need some mathematical assistance also. So soon as it becomes possible to construct biological theories cast in mathematical form we may look for rapid progress. Experiment will then be guided along those lines where it is most apt to yield new knowledge of importance. This has been the history of physics and chemistry. The evolution of chemistry into an exact science has been so recent that many of us have actually witnessed the events. We can hardly doubt that a similar outcome for the biological sciences will one day become a matter of history also.

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