ON THE INTEGRATION IN FINITE TERMS OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER*

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1. *Introduction.* Liouville, in 1840, investigated the equation

$$\frac{d^2w}{dz^2} = \chi(z)w, \tag{A}$$

with $\chi(z)$ an elementary function of z, to determine under what circumstances the solutions of the equation are elementary.†

By an "elementary function", Liouville understood, in this connection, any function of z obtained in a finite number of steps by performing algebraic operations, taking logarithms and exponentials, and performing integrations. An example of such a function would be

$$\int\left[\log \arcsin z + \int z^2dz\right]^{1/2} dz + \tan \log_z[1 + (z)^{1/2}].$$

It is a consequence of Liouville's work that the solutions of Bessel's equation

$$z^2w'' + zw' + (z^2 - \nu^2)w = 0$$

are elementary functions only when 2ν is an odd integer.‡ But it would be improper to conclude from Liouville's results alone, as some authors seem to do, that Bessel's equation cannot be solved by the simpler formal methods of the theory of differential equations. For instance, the equation

$$\frac{du}{dz} = \frac{u}{u-1}$$

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† Journal de Mathématiques, vol. 5 (1840). See also Watson, *Theory of Bessel Functions*, Cambridge, 1922, p. 111. No acquaintance with Liouville's work is necessary for the understanding of the present paper.

‡ Watson, loc. cit.

integrates directly into

$$u - \log u = z + c,$$

but the functions u thus obtained cannot be expressed in terms of z by performing algebraic operations and taking exponentials and logarithms. Can one be sure in advance that Bessel's equation is not susceptible to similar treatment?

Investigations on the possibility of integrating differential equations of the first order by elementary operations performed upon both the dependent variable and the independent variable were carried out by Lorenz, Steen, Hansen and Petersen,* and by Mordukhai-Boltovskoi.† Lagutinski has studied systems of equations of the first order.‡

In the present paper, we prove a general theorem (§3) which shows that the solutions of Bessel's equation do not satisfy elementary equations,§ except for the given values of ν.

Our theorem states that if any solution of equation (A) satisfies an elementary equation in w and z, then the general solution of (A) is an elementary function of z.

The procedure in the present paper is purely formal. The many questions of a function-theoretic nature which will arise will be found treated in our papers on elementary functions published in the Transactions of this Society.‖

2. *The l-Functions.* We shall call any algebraic function of w and z an *l-function of order zero*, and the variables w and z *monomials of order zero*.

We introduce now two classes of *monomials of order one*. The first class will consist of the exponentials of non-constant algebraic functions of w and z. The second will consist of all

* Nyt Tidskrift for Matematik, 1874–1876.

† Communications de la Société Mathématique de Kharkow, vol. 10 (1909), p. 34.

‡ Kharkow Technological Institute, 1909. I have not been able to secure Lagutinski's paper, but the abstract in the Jahrbuch indicates that the present paper does not overlap on it.

§ This term, whose meaning is fairly clear from what precedes, is made precise below.

‖ Vol. 25 (1923), p. 211, and vol. 27 (1925), p. 68.

functions which are not algebraic, but whose partial derivatives are both algebraic. Evidently the logarithm of any non-constant algebraic function is a monomial of the second class.

By an *l-function of order one*, we shall mean any non-algebraic function u of w and z which is obtained by performing algebraic operations upon monomials of orders zero and one; that is, a function which satisfies an equation

$$\alpha_0(w,z)u^n + \alpha_1(w,z)u^{n-1} + \cdots + \alpha_n(w,z) = 0, \tag{1}$$

in which every $\alpha(w, z)$ is a rational integral combination of monomials of orders zero and one.

We now define, by induction, l-functions of any order n. A function which is not among the functions of order 0, 1, $\cdots$, $n-1$, will be called a *monomial of order* n if it is of one of the following two types:

(a) an exponential of a function of order $n-1$;

(b) a function both of whose partial derivatives are among the functions of order 0, 1, $\cdots$, $n-1$.

Evidently a logarithm of a function of order $n-1$ is a monomial of order n if it is not a function of order less than n.

Any function u will be called an *l-function of order* n if it is not an l-function of order less than n, and if it satisfies an equation like (1) in which each $\alpha(w, z)$ is a rational integral combination of monomials of orders 0, 1, $\cdots$, n.*

The result of performing algebraic operations upon l-functions of the first n orders is always an l-function of one of those orders.

The partial derivatives of any l-function u of order n are both l-functions of order n or less. If, in the expression for u, the maximum of the orders of the monomials which involve w is r, then the derivatives have expressions for which the same maximum is at most r, and in which no monomials of order r involving w appear which do not also appear in u. These facts are a consequence of the rules for differentiation.

* It is not important here to know whether functions of every order actually exist. See Liouville, Journal de Mathématiques, vol. 3 (1838), p. 523.

3. *Linear Differential Equations.* We prove the following theorem, which shows that the solutions of Bessel's equation do not satisfy an elementary equation, except for the stated values of ν.

THEOREM. *If w, any solution of the differential equation $w''=\chi(z)w$, in which $\chi(z)$ is an l-function, satisfies an equation $F(w, z)=0$, where $F(w, z)$ is an l-function, not identically zero, then w is an l-function of z.*

Differentiating the equation $F(w, z)=0$ with respect to z, we find

$$F_w w' + F_z = 0. \tag{2}$$

Here F_w and F_z are l-functions. We have thus, in the first member of (2), an expression algebraic in w', and in l-functions of w and z, which vanishes when w is the given solution of $w''=\chi(z)w$, *without vanishing identically.**

With every such expression, $f(w', w, z)$, we associate two integers; first r, the maximum of the orders of those monomials in w and z, contained in the expression, which involve w; secondly s, the number of such monomials of order r which involve w. Many different expressions will represent the same function; the integers will vary with the expression.

There is a class of expressions $f(w', w, z)$ for which r is a minimum, and in this class there are expressions for which s is as small as it can be if r is a minimum. We assume that we have in hand an $f(w', w, z)$ with r and s minima, and write

$$f(w', w, z) = 0. \tag{3}$$

We are going to prove that $r=0$. Let the contrary, namely that $r>0$, be assumed, and let θ be one of the monomials of order r contained in f which involve w. Since f involves θ algebraically, we find from (3), upon solving for θ,

$$\theta - g(w', w, z) = 0, \tag{4}$$

* Of course, $F(w, z)$ is also such an expression, but (2) gives a better example.

where g is an algebraic function of w' and of l-functions of w and z, with $s-1$ monomials of order r which involve w.

If θ is an exponential, e^v, where v is of order $r-1$, we write (4) in the form

$$v - \log g(w', w, z) = 0. \tag{5}$$

If θ is an integral, that is, a monomial of the second class, we let (4) stand. We let the first member of the equation

$$G(w', w, z) = 0 \tag{6}$$

represent the first member of (4), or that of (5), according as θ is an integral or an exponential.

On differentiating (6) we find, remembering that $w'' = \chi(z)w$,

$$\chi(z)wG_{w'} + G_w w' + G_z = 0. \tag{7}$$

Now if the first member of (7) did not vanish identically with respect to w', w and z, we would have an equation of the form (3) involving at most $s-1$ monomials of order r which contain w. For, in (4) and (5), the partial derivatives of θ and v are of order less than r, and the derivatives of g and $\log g$ contain at most $s-1$ monomials of order r involving w.* Hence the first member of (7) vanishes identically.

Let μ be any constant. If w is the given solution of the equation $w'' = \chi(z)w$, the derivative with respect to z of

$$G(\mu w', \mu w, z) \tag{8}$$

will evidently be the first member of (7) with w' and w replaced by $\mu w'$ and μw.† As (7) holds for arbitrary values of the variables, the derivative just mentioned must vanish for every z. Hence the function of (8) depends only on μ, and not on z. We write

$$G(\mu w', \mu w, z) = \beta(\mu). \tag{9}$$

* Final remarks of §2, which obviously apply also to $G_{w'}$.

† The device now being used is analogous to, though not identical with, one which underlies Liouville's work on the elementary functions. We use below a method given by us in the Comptes Rendus for Aug. 21, 1926, which permits a great simplification of the work of Liouville and his followers.

We differentiate (9) partially with respect to μ, and then put $\mu=1$. Writing $\beta'(1)=a$, we have

$$w'G_{w'} + wG_w = a, \tag{10}$$

and, as in the case of (7), we see that (10) is an identity in w', w and z.

A particular solution of (10) is $G=a \log w$, and w'/w is a solution of the equation obtained on replacing the second member of (10) by zero. Hence

$$G = a \log w + H\left(\frac{w'}{w}, z\right). \tag{11}$$

As G is algebraic in w', H must be an algebraic function of w'/w and of l-functions of z. By (6) and (11), we have, for w as in the hypothesis,

$$a \log w + H\left(\frac{w'}{w}, z\right) = 0. \tag{12}$$

If $a=0$, we have a contradiction of the assumption that $r>0$.

Suppose that $a\neq 0$. Writing $u=w'/w$ and differentiating (12), we find

$$au + [\chi(z) - u^2]H_u + H_z = 0. \tag{13}$$

This equation must be an identity in u and z, else it would be an equation (3) with $r=0$. But we shall show that no H which involves u algebraically can satisfy (13), thus making untenable, finally, the assumption that $r>0$.

If H is algebraic in u, any of its branches has, for the neighborhood of $u=\infty$, a development in descending powers of u,

$$H = \alpha_1(z)u^{p_1} + \alpha_2(z)u^{p_2} + \cdots + \alpha_n(z)u^{p_n} + \cdots, \tag{14}$$

where the decreasing exponents p_n are rational, and all have the same denominator, which may be unity. We may, and shall, assume that no $\alpha(z)$ is zero for every z.

If $p_1\neq 0$, the development of $(\chi-u^2)H_u$ begins with a term in u^{p_1+1}, whereas, in H_z, the first exponent does not exceed p_1. Hence the first term of $(\chi-u^2)\, H_u$ must balance with au

in (13), an impossibility if $p_1 \neq 0$. If $p_1 = 0$, $(\chi - u^2)H_u$ begins with u^{p_2+1}, and H_z begins with a zero or negative power. Now, as $a \neq 0$, we have to balance au in (13). This is impossible, because $p_2 + 1 < 1$. Thus no H algebraic in u satisfies (13).

We have proved that $r = 0$ for the first member of (3). If, now, (3) does not involve w', then, since it is algebraic in w, we can solve for w, finding w to be an l-function of z. If (3) does involve w', we solve for w', finding

$$w' = H(w, z), \tag{15}$$

where H is an algebraic function of w, and of l-functions of z.

On differentiating (15), remembering that $w'' = \chi(z)w$, we find

$$\chi(z)w = H_w H + H_z. \tag{16}$$

If (16) is not an identity in w and z, w is determined as an l-function of z. Let, then, (16) be an identity, and consider, for the neighborhood of $w = \infty$, a development of H in descending powers,

$$H = \alpha_1(z)w^{p_1} + \alpha_2(z)w^{p_2} + \cdots + \alpha_n(z)w^{p_n} + \cdots .$$

As the coefficients α are found by algebraic operations from the equation which H satisfies together with w and l-functions of z, the α 's are all l-functions.

When we substitute the development of H into (16) ,we find that $p_1 = 1$ and that

$$\alpha_1'(z) + [\alpha_1(z)]^2 = \chi(z).$$

Thus, $\alpha_1(z)$ is a solution of the Riccati equation $u' + u^2 = \chi(z)$, obtained from $w'' = \chi(z)w$ by putting $u = w'/w$. Hence the exponential of the integral of $\alpha_1(z)$, which is an l-function not identically zero, satisfies the equation $w'' = \chi(z)w$.

But it is well known, and easy to see, that when a single solution of a linear homogeneous equation of the second order is known, the general solution can be found from it by a quadrature. Hence every solution of (A) is an l-function, and the theorem is proved.

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