A THEOREM CONCERNING DIRECT PRODUCTS*

BY LOUIS WEISNER†

The theorem in question may be stated as follows. *A group of order mn, where m and n are relatively prime, in which every element whose order divides m is commutative with every element whose order divides n, is the direct product of two groups of orders m and n.*

Burnside‡ has proved the theorem for the special case in which either m or n is a power of a prime. Hence, to prove the theorem, we need only show that it is true for groups of order mn if it is true for groups of order $<mn$. To avoid trivial cases we assume that m and $n>1$. We denote by m_1, m_2, and n_1, n_2, divisors >1 of m and n respectively.

If the group G contains an element of order m, let p be a prime factor of m and p^α the highest power of p that divides m. Every element of G whose order divides p^α is commutative with every element whose order is prime to p. It follows from the special case referred to, that G contains an invariant subgroup of order mn/p^α, which contains an invariant subgroup of order n, as every element whose order divides m/p^α is commutative with every element whose order divides n.

We suppose now that G contains no element of order m. The normaliser H of an element s of order m_1 includes all the Sylow subgroups of G whose orders divide n, and is therefore of order m_2n, where $m_2 \geqq m_1$. If $m_2<m$, H contains an invariant subgroup of order n. If $m_2=m$, s is invariant under G. An element t of G corresponding to an element t' of $G/(s)$ of order n_1 is of order $n_1\mu$, where μ divides m_1. Hence G contains§ two elements t_1 and s_1 of orders n_1 and μ respectively, such that

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‡ W. Burnside, *Theory of Groups*, 2d edition, p. 327.

§ Loc. cit., p. 16.

$t=t_1s_1$. Since $(t_1s_1)^{n_1}=s_1^{n_1}$ is in (s), and n_1 is prime to the order of s_1, s_1 is a power of s. Thus t_1, as well as t, corresponds to t' in the isomorphism of G with $G/(s)$; but t_1 and t' are of the same order n_1. Every element of $G/(s)$ whose order is a divisor of m/m_1 corresponds to an element of G whose order is a divisor of m. It follows that t', and hence every element of $G/(s)$ whose order divides n, is commutative with every element whose order divides m/m_1. Hence $G/(s)$, being of order $<mn$, contains an invariant subgroup of order n. The corresponding subgroup of G, being of order $m_1n<mn$, also contains an invariant subgroup of order n.

Thus in all cases G contains a subgroup of order n. Similarly G contains a subgroup of order m. G is evidently the direct product of these two subgroups.

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A CUBIC CURVE CONNECTED WITH TWO TRIANGLES

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1. *Introduction.* If ABC, XYZ are two triangles, a cubic curve Γ_3 may be associated with them as follows.* Let (PQ, RS) denote the point of intersection of the lines PQ, RS; then Γ_3 is the locus of a point O such that (OA, YZ), (OB, ZX), (OC, XY) are collinear and also the locus of a point O for which (OX, BC), (OY, CA), (OZ, AB) are collinear. In fact when one set of three points is collinear the other set of three is also collinear. Take ABC as triangle of reference and let the points X, Y, Z have coordinates (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) respectively, then if (α, β, γ) are current coordinates

* H. Grassmann, *Die lineale Ausdehnungslehre*, 1844, p. 226. The corresponding quartic surface connected with two tetrahedra is mentioned by H. Fritz, Pr. Ludw. Gymn. Darmstadt [reference taken from Jahrbuch der Fortschritte der Mathematik, vol. 21 (1889), p. 725] and by C. M. Jessop, *Quartic Surfaces*, Cambridge, 1916, p. 189.