ON THE FUNCTIONAL EQUATION $f(x+y)=f(x)+f(y)$*

BY MARK KORMES

Fréchet,† and later Blumberg‡ and Sierpinski,§ have demonstrated that the solution of the functional equation

$$f(x+y)=f(x)+f(y) \tag{1}$$

which is measurable, has the form $A\cdot x$, where A denotes a constant. In this note the following theorem is proved.

THEOREM I. *Every solution of the functional equation* (1) *which is bounded on a set of positive measure is of the form* $A\cdot x$.

The proof depends on a theorem of Steinhaus¶ which can be stated as follows.

LEMMA. *The set arising by arithmetic summation* (*addition of abscissas*) *of a set of positive measure, contains an interval.*||

Since $f(x)$ is bounded on a set of positive measure, $f(x+y)$ is bounded on an interval, and therefore $f(x)$ must be of the form $A\cdot x$ according to a theorem of Darboux.

THEOREM Ia. *The statement of Theorem* I *remains true if* $f(x)$ *is bounded on a set whose interior measure is positive.*

If the interior measure of a set A is $a>0$, then there exists** a measurable sub-set of A whose measure is equal to a (>0).

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† M. Fréchet, L'ENSEIGNEMENT MATHÉMATIQUE, vol. 15 (1913), p. 390.

‡ Blumberg, *Convex functions*, TRANSACTIONS OF THIS SOCIETY, vol. 20, p. 41.

§ W. Sierpinski, FUNDAMENTA MATHEMATICAE, vol. 1, p. 116.

¶ H. Steinhaus, FUNDAMENTA MATHEMATICAE, vol. 1, p. 99.

|| The proof of this lemma will be a part of a paper entitled *On arithmetic summation of point sets.*

** C. Carathéodory, *Vorlesungen über Reelle Funktionen*, p. 261.

To this subset Theorem I can be applied and thus Theorem Ia is established.

Theorem I establishes a far more general condition than the one given by Fréchet, Blumberg, and Sierpinski. The following remarks show that this condition is incisive. The condition $m_i(M)^* > 0$ is essential, since there exist non-measurable solutions of (1) which are continuous on a set H, where, for every interval δ,

$$m_i(H \cdot \delta) = 0, \qquad m_e(H \cdot \delta) = \delta.$$

Let B denote a hamelian basis-set of all real numbers. If b is a number of B, we define a solution of the functional equation (1) as follows:

$$f(x) = 0, \quad \text{for the numbers of the set } (B - b);$$
$$f(x) = 1, \quad \text{for } x = b;$$
$$f(x+y) = f(x) + f(y), \quad \text{for all real numbers.}$$

In this way $f(x)$ is completely defined. Let us denote by H the set of all points where $f(x) = 0$. If we denote by H^c the set of all numbers $x+c$, where x assumes all values of H we have then $H \cong H^{\alpha b}$, where the symbol $\cong$ means congruent, and α is a rational number. Then we have

$$(H^{\alpha b} \cdot H^{\alpha' b}) = 0,$$

if $\alpha \neq \alpha'$, and

$$K = \sum_{\alpha} H^{\alpha b},$$

where K denotes the continuum. Therefore we must have,† for every interval δ,

$$m_i(H \cdot \delta) = 0, \qquad m_e(H \cdot \delta) = \delta.$$

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* The symbol m_i (M) shall signify the interior measure of M, $m_e(M)$ the exterior measure of M.

† For suppose $m_i(H \cdot \delta) > 0$. There must exist then a measurable subset $P \subset H$ so that $m(P) > 0$. We would have $m(P^{\alpha b}) = m(P) > 0$. On the other hand it can be shown easily, that then there exists a rational number α_1, so that $m(P^{\alpha_1 b} \cdot P) = a > 0$, but this is impossible, since $(P^{\alpha_1 b} \cdot P) = 0$ because $P \subset H$, $P^{\alpha_1 b} \subset H^{\alpha_1 b}$, and $(H \cdot H^{\alpha_1 b}) = 0$. We must have therefore $m_i(H \cdot \delta) = 0$. See also M. Kormes, *Treatise on basis-sets* (Columbia University dissertation, not yet published), Theorem VIII.

Since the function $f(x)$ is everywhere 0 on the set H, it is bounded and continuous.

There exist non-measurable solutions of (1), which are continuous on a perfect set P, where $m(P)=0$. Let P be the set of all numbers z of the form

$$z = \frac{x_1}{10x_1!} + \frac{x_2}{(10^2x_2 + 10x_1)!} + \cdots + \frac{x_n}{(10^n x_n + 10^{n-1}x_{n-1} + \cdots + 10x_1)!} + \cdots,$$

where every x_n is either 1 or 2. There cannot exist then any relation of the form

$$\sum_\lambda r_\lambda z_\lambda = 0$$

between the numbers z of the set P,* where r_λ denotes a rational number, and in every case only a finite number of r_λ are different from 0. The numbers of P constitute a subset of a basis-set B of all real numbers.† The existence of such basis-set was demonstrated in another paper.‡

We define now a solution of the functional equation (1) in the following way:

$f(x)=0$, for all numbers of P;

$f(x)=1$, for all numbers of $B-P$;

$f(x+y)=f(x)+f(y)$ for all numbers of the continuum K. But this defines $f(x)$ completely, and it is clear that $f(x)$ is non-measurable and continuous on the perfect set B.

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* M. Kormes, *Treatise on basis-sets.*

† To construct a basis-set B which has a given set P as a subset we proceed in the following way. We well-order the continuum K in such a way that the numbers of P precede all other numbers. The set $(K-P)$ is not empty, and since the set P is not the entire basis-set of K, there must be a first number a_1 of $(K-P)$ which cannot be represented by numbers of P in a linear way. If we consider the set $P_1=P+a_1$ and reason in the same way as above, we obtain a basis-set B of the continuum K. See also M. Kormes, *Treatise on basis-sets.*

‡ M. Kormes, loc. cit.

From Theorem I, the Fréchet-Sierpinski theorem* can be deduced immediately.

THEOREM II. *Every solution of* (1) *which is measurable has the form* $A \cdot x$.

In fact, suppose that $f(x)$ is a solution of (1), and that $f(x)$ is measurable. Then there exists a perfect set P, where $m(P)>0$, and $f(x)$ is continuous on P. Being finite, $f(x)$ must be bounded on P, and Theorem II is a simple consequence of Theorem I.

Theorem I can be generalized for functional equations in n variables. A proof for two variables will be given below and it is quite analogous for $n(>2)$ variables.

THEOREM III. *Every solution of the functional equation*

$$f(x+u,\ y+v) = f(x,y) + f(u,v) \tag{2}$$

where x, y, u, v *denote real numbers, which has the property that* $f(x,\ 0)$ *is bounded on a measurable set* M_x, *where* $m(M_x)>0$, *and that* $f(0,\ y)$ *is bounded on a measurable set* M_y, *where* $m(M_y)>0$, *has the form* $A \cdot x+B \cdot y$.†

We have

$$f(x,y) = f(x+0,\ 0+y) = f(x,0) + f(0,y),$$

where $f(x,\ 0)$ is the solution of the functional equation

$$f(x+u,\ 0) = f(x,0) + f(u,0),$$

and $f(0,\ y)$ is the solution of the functional equation

$$f(0,\ y+v) = f(0,y) + f(0,v).$$

According to Theorem I, $f(x,\ 0)$ has the form $A \cdot x$, where $A=f(1,\ 0)$; and $f(0,\ y)$ has the form $B \cdot y$, where $B=f(0,\ 1)$. Therefore $f(x,\ y)$ has the form $A \cdot x+B \cdot y$.

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* See second, third, and fourth footnotes on p. 689.

† We can assume also that $m_i\ (M_x)$ and $m_i\ (M_y)>0$, and reason in a way similar to that indicated in the proof of Theorem Ia.

The same reasoning holds if $f(x, a)$ and $f(b, y)$ are bounded in M_x and M_y, respectively. We have then $f(x, a) = f(x, 0)+f(0, a)$ and $f(b, y)=f(0, y)+f(b, 0)$. Hence $f(x, 0)$ and $f(0, y)$ must be therefore bounded in M_x and M_y, respectively.

From Theorem III, the following theorem can easily be obtained.

THEOREM IV. *Every solution of the functional equation* (2) *which is bounded on a measurable set M_{xy}, whose square measure is $m^{(2)}(M_{xy})>0$, has the form $A \cdot x+B \cdot y$.*

In order to prove this theorem let us suppose that

$$m^{(2)}(M_{xy}) = a > 0.$$

According to a theorem of Fubini,* there must exist then a straight line $y=b$ parallel to the X-axis, and a straight line $x=a$ parallel to the Y-axis, so that $m(M_{xa})>0$ and $m(M_{by})>0$. Then $f(x, a)$ would be bounded on the set M_{xa}, where $m(M_{xa})>0$; and $f(b, y)$ would be bounded on M_{by}, where $m(M_{by})>0$. Therefore $f(x, y)$ must have the form

$$A\cdot x + B\cdot y.$$

From Theorem IV, we may state the following theorem.

THEOREM V†. *Every solution of the functional equation* (2) *which is measurable has the form $A \cdot x+B \cdot y$.*

If $f(x, y)$ is a solution of (2) and it is measurable, then there exists a perfect set P, where $m^{(2)}(P)>0$, and $f(x, y)$ is measurable on P. Since $f(x, y)$ is finite and P is closed, $f(x, y)$ is bounded on P, and we can apply Theorem IV.

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* Theorem of Fubini-Lebesgue; see de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, vol. 2 (2d ed.), pp. 117–120.

† Theorem of Steinhaus-Sierpinski; see Sierpinski, loc. cit,