A THEOREM ON CONNECTED POINT SETS WHICH ARE CONNECTED IM KLEINEN*

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It has been shown by Miss Mullikin† that if K and M are two closed,‡ mutually exclusive point sets and H is a closed, bounded, connected set having at least one point in common with each of the sets K and M, then there exists a point set L, a subset of H, such that L is connected and contains no point of either K or M, but such that K and M each contain at least one limit point of L.

If H is not closed, the above theorem no longer holds, as can be shown by very simple examples. It is the purpose of this note to establish an analogous theorem for the case where H, although not closed, is connected im kleinen.§

THEOREM. *Let K and M be two closed mutually exclusive point sets and N a connected, connected im kleinen point set*

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† *Certain theorems relating to plane connected point sets*, TRANSACTIONS OF THIS SOCIETY, vol. 24 (1922), pp. 144–162. Rosenthal gave a proof for that case where each of the sets K and M reduces to a single point. See A. Rosenthal, *Teilung der Ebene durch irreduzible Kontinua*, MÜNCHENER SITZUNGSBERICHTE, MATHEMATISCH-PHYSIKALISCHE KLASSE, 1919, p. 104.

‡ A point set is said to be (1) *closed*, if it contains all its limit points; (2) *connected*, if it is not the sum of two mutually exclusive points sets neither of which contains a limit point of the other; (3) *bounded*, if it lies entirely in a finite portion of the space under consideration.

§ A point set M is said to be *connected im kleinen* at a point P if for every circle K_1 with center at P there exists a concentric circle K_2 such that every point x of M which lies interior to K_2 is joined to P by a connected subset of M which lies wholly within K_1. M is itself said to be connected im kleinen if it is connected im kleinen at every point. See Hans Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, WIENER SITZUNGSBERICHTE, vol. 123, Abt. IIa (1914), pp. 2433–2489; also *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, JAHRESBERICHT DER VEREINIGUNG, vol. 23 (1914), pp. 318–322. See also S. Mazurkiewicz, *Sur les lignes de Jordan*, FUNDAMENTA MATHEMATICAE, vol. 1 (1920), pp. 166–209, and earlier papers in Polish referred to therein.

which has at least one point in common with each of the sets K and M. Then there exists a point set H, a subset of N, such that H is connected and contains no point of either K or M, but such that K and M each contain at least one point of N which is a limit point of H.

PROOF. Let

$$N \times K = k,$$
$$N \times M = m.$$

For every point P of $N-(k+m)$, consider the composant* $C(P)$ determined by P in that set.

No $C(P) \equiv P$. For there exists a circle, C, with center at P, which encloses no point of the closed set $K+M$. As N is connected and connected im kleinen, there exists at least one point x of N, distinct from P, which lies, with P, in a connected subset of N which lies wholly interior to C and can, therefore, contain no point of $k+m$. Hence x is a point of $C(P)$ distinct from P.

Every $C(P)$ has a limit point in k or m. For suppose not. Then let x be a point such that $C(x)$ has no limit point in $k+m$. Then

$$N = C(x) + N',$$

where N' consists of all points of N not contained in $C(x)$.

Either $C(x)$ contains a limit point of N' or vice versa. Let y be a limit point of N' in $C(x)$. As $K+M$ is closed, there exists a circle C with center at y and enclosing no point of $K+M$. As N is connected im kleinen there exists a point z of N' interior to C which lies with y in a connected subset of N which lies wholly interior to C and hence contains no point of $k+m$. Then z is a point of $C(x)$. This is impossible. Hence $C(x)$ cannot contain any limit point of N'. Similarly N' cannot contain any limit point of $C(x)$. Hence N is the sum of two mutually exclusive sets neither of which contains a limit point of the other. But this is impossible

* I.e., the set of all points which lie with P in a connected subset of $N-(k+m)$.

since N is connected. Thus the supposition that a $C(P)$ exists which has no limit point in k or m leads to a contradiction.

If any $C(P)$ has limit points in both k and m the theorem is proved. Suppose this is not the case.

Separate N into two sets, N_1 and N_2, such that

$$N_1 = k + Q_1 \quad \text{and} \quad N_2 = m + Q_2$$

where Q_1 contains all points x of $N-(k+m)$ such that $C(x)$ has a limit point in k, and Q_2 contains all points x of $N-(k+m)$ such that $C(x)$ has a limit point in m.

As N is connected, N_1 contains a limit point of N_2, or vice versa. Suppose N_2 contains a limit point, t, of N_1.

The point t cannot belong to Q_2, since in this case it could be shown that t is joined to some point of Q_1 by a connected subset of $N-(k+m)$, by virtue of the connectedness im kleinen of N. Hence t must be a point of m.

Let T be a circle with center at t and enclosing no point of K. There exists, because of the connectedness im kleinen of N, a connected subset, R, of N, which contains some point s of Q_1 and t, and lies wholly interior to T. Let

$$[C(s)+R]-C(s)=R'.$$

The set $C(s)+R'$ is connected. Now $C(s)$ has no limit point in R' since such a point would belong to the set $N-(k+m)$ and hence also to the set $C(s)$. On the other hand, if R' has a limit point, u, in $C(s)$, it can be readily shown by application of the connectedness im kleinen of N at u that at least one point of R' belongs to $C(s)$, which is of course impossible. Then $C(s)+R'$ is the sum of two mutually exclusive sets neither of which contains a limit point of the other, which contradicts the fact that $C(s)+R'$ is connected.

Thus the supposition that no $C(P)$ has a limit point in both k and m leads to a contradiction and the theorem is proved.

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