# SOME MODERN VIEWS OF SPACE*

BY JAMES PIERPONT

1. *Introduction.* We are living in an age of great discoveries; in physics, in chemistry, in astronomy; in the field of invention one is almost bewildered by the great achievements which have been made in recent years. Although figuring less in the public eye, the development of mathematics has been no less remarkable. In the present paper I wish to outline briefly what progress has been made in our knowledge of space from a mathematical standpoint.

Until about a century ago everybody believed that the geometry of Euclid gave an exact description of space as far as it went. Geometry as the science of space has to deal with points, straight lines and planes. What are these things? Euclid says: A point is that which has no part, a line is breadthless length, a straight line is a line which lies evenly with the points on itself, and so on. I do not need to continue. As we see, these definitions would not tell one what a point, a straight line, a plane, are if one did not already have these notions in his mind. Euclid probably did not intend that they should be regarded otherwise than briefly describing some of their salient properties.

We see on all sides of us lines which are approximately straight, and surfaces which are approximately plane. A stretched string or a ray of light visualize a straight line, and the surface of a pond a plane. In machinery plane surfaces are of great importance; the engineer, the physicist and the astronomer are vitally interested in them. How are they constructed? One takes three metal plates nearly plane and rubs them pairwise together using some abrasive powder. In this way we get slightly spherical surfaces, one concave and two convex or two concave and one convex.

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By rubbing surfaces together of opposite curvature we flatten them. When nearly flat the surfaces may be scraped. Their flatness is tested by applying a fine layer of pigment and then rubbing them together. Plates may be prepared in this way so flat that, although one surface slides freely over the other, it requires considerable force to pull them apart. In optics, especially for astronomical needs, plane glass surfaces are ground so flat that they do not deviate from true flatness by more than a tenth of a wave length of light, say one five-hundred-thousandth of an inch.

In this way we may prepare three right trieders each formed of three plane surfaces such that two of them resting on a third, touch perfectly along another pair of their surfaces. Such plane surfaces cut at right angles; their three edges are straight lines which meet in a point. We are now in position to construct a straight edge or ruler and hence to draw straight lines; also to construct right triangles. With straight edge and triangle we can draw parallels. By means of dividers or compasses we can draw circles and lay off equal segments and angles. With this small equipment we can test some of the propositions of euclidean geometry by actual construction. Some, by the means we here employ, we can never verify. For example, Euclid says only one parallel can be drawn to a given straight line. Obviously on any given drawing board we can draw through a given point a great many lines which do not cut a given line on the board. In our mind we enlarge the board, yet for any board however large, we see at once that there are always a great many non-cutting straight lines. Only when we take an infinite plane can we assert that there is even one parallel. But now that we have an infinite plane, how do we know that there may not be more than one? What do we know about the infinite part of space anyway?

We may go further and ask: Is there an infinite part of space? To the naive mind this last question seems almost foolish. Is not space boundless? Is there not always a beyond?

There are three main ways in which we may answer these questions.

1) Space is infinite and there is only one parallel to a given straight line through a given point.

2) Space is infinite and there are two parallels to a given line.

3) Space is boundless but not infinite; there are no parallels since any two straight lines in a plane always cut.

The first gives us the geometry of Euclid, the geometry of our every day life, the geometry on which astronomy, physics, and engineering rest or did rest till the advent of Einstein. The second answer was adopted by Lobachevsky and Bolyai just about a century ago (1823–33). That the third answer is equally possible was first established by Riemann about thirty years later (1854) although his results were not published until 1868. The second and third geometries are called non-euclidean. All three depend upon a certain constant which we may call the *space constant* $k$. In the geometry of Euclid $k$ is zero, in the geometry of Lobachevsky $k$ is negative, in the third geometry $k$ is positive. In the classification of conics a similar distinction arises, and this has led Klein to call these geometries *parabolic*, *hyperbolic*, and *elliptic*, respectively.

The reception which greeted the epoch making discoveries of Lobachevsky and Bolyai was cold indeed. Geometers at that time like a great part of the learned world lay supinely under the spell of the great Königsberg philosopher Kant, according to whom geometry, that is euclidean geometry, is an a priori science having apodictic certitude. Gauss years before (1792–1817) had discovered the hyperbolic geometry, but had not dared to publish his results.

Writing to Gerling* (1818) he says "I am glad that you have the courage to (publically) acknowledge the possibility that our theory of parallels and hence our whole geometry may be false." "But the wasps whose nest you thus disturb will

* Gauss, WERKE, vol. 8, p. 179.

fly about your head." In 1829 he confesses to Bessel* that his dread of the clamor of the Boeotians, that is the adherents of Kant, may prevent him ever during his life time from publishing his extensive researches regarding the foundations of geometry. Like Newton, Gauss preferred his peace of mind to glory if it had to be purchased at the price of endless wrangling. After his death the world learned Gauss' views through the publication (1860–65) of his correspondence with Schumacher and in the biographical sketch by his friend Baron von Waltershausen (1856). Influenced by the great weight of Gauss' name, geometers directed their attention once more to non-euclidean geometry. In 1868 Beltrami published his representation of the hyperbolic plane on the pseudosphere; in the GÖTTINGER NACHRICHTEN of the same year appeared Helmholtz's paper on the foundation of geometry, while in the ABHANDLUNG of the Göttingen Royal Society of this year was published the epoch making paper of Riemann mentioned above. In 1871 Klein, taking over some of Cayley's results, showed how to establish non-euclidean geometry by projective methods. From now on, labor in these fields of research has never halted. With the advent of Einstein's general theory of relativity (1914–16) a new epoch began. Before we touch on this I wish to note briefly a few of the outstanding facts of non-euclidean geometry.

2. *Non-Euclidean Geometry.* Suppose for the moment our space were non-euclidean. We would have rigid bodies in it which could be moved about freely without any distortion, just as in euclidean space. A stretched string and a ray of light would be straight lines in this geometry. The method of preparing plane surfaces and right trieders described above would still hold good. Everything immediately about us would behave as far as we could ascertain as if space were euclidean. Let $a$, $b$, $c$ be the lengths of the three sides of a

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* Ibid., p. 200; "da ich das Geschrei der Böotier scheue."

triangle and $A$, $B$, $C$ the angles opposite these sides. Then our theory tells us:

$$(1)\quad \begin{cases} \sin A : \sin B : \sin C = a : b : c & \text{in euclidean space;} \\ \sin kA : \sin kB : \sin kC = a : b : c & \text{in elliptic space;} \\ \sinh kA : \sinh kB : \sinh kC = a : b : c & \text{in hyperbolic space.} \end{cases}$$

Since as far as our measurements go, the first relation holds, the space constant $k$ is small.

To picture to our minds how large portions of our space would look if non-euclidean, we may imitate the geographers who represent the spherical surface of the earth on a plane by different kinds of maps. As is well known, the shortest path joining two points on a sphere is an arc of a great circle, or a *geodesic*; and these take the place of straight lines in a plane. I would like to call your attention to two maps, known as the stereographic and central projections. In the first, geodesics are represented by circles; in the second by straight lines. Stereographic maps are conformal, i.e., two intersecting curves meet on the map under the same angle as on the sphere. In both maps distances are distorted.

We have similar representations or models of a non-euclidean space in a euclidean space, with this difference: if we wish to represent our non-euclidean straight lines by euclidean straights, our model will preserve neither angles nor lengths. We shall therefore speak only of the model which preserves angles. We begin with hyperbolic space.

3. *Hyperbolic Space.* We take an $e$-sphere $S$ of radius $R$, which we call the fundamental sphere.* All points of $H$-space lie within $S$. $H$-straights are $e$-circles cutting $S$ orthogonally; these are the paths of light in this space. $H$-planes are $e$-spheres cutting $S$ orthogonally. Hence two $H$-planes cut in an $H$-straight. If our space were hyperbolic our whole universe would lie within $S$ and to bodies moving about in $e$-space would correspond figures moving about within $S$. Any one who has looked into a convex mirror (common

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* For $e$, read *euclidean*; for $H$, read *hyperbolic*; for $E$, read *elliptic*.

enough on automobiles now) will grasp the spirit of this model. Suppose one wished to measure an $H$-straight (Fig. 1) cutting $S$ in $A$ and $B$, starting from a point $P$ on it. His measuring rod in the model would get shorter and shorter as he moves toward $A$ or $B$ so that $PA$ and $PB$ would have an infinite length. Any point $Q$ within $S$ is at an infinite distance in $H$-measure from any point on $S$. No displacement in space takes a point on $S$ in the model off from $S$. Let $ACD$ in Fig. 1 be an $H$-straight $l$ meeting $S$ at $A$ and $D$, and $P$ a point without. Let the $H$-straights $AP$, $DP$ cut $S$ in $B$ and $E$. Let an $H$-straight $m$ rotate about $P$ in the plane of $P$, $l$.

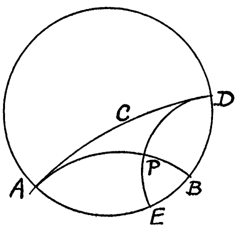

FIG. 1

When $m$ lies within the sector $APD$, it will cut $l$; when $m$ lies within the sector $DPB$ it will not cut $l$. The two $H$-straights $PD$ and $PA$ are the limiting positions of straights through $P$ separating the straights which cut $l$ from those which do not. They are the parallels of $H$-geometry. In $H$-space the sum of the angles of any plane triangle $ABC$ is less than two right angles. This is obvious in our model. For since figures may be moved without distortion, we may move the plane of $ABC$ so that $A$ coincides with the center of $S$. Two of the sides in the model are now $e$-straights while the third side $BC$ is an $e$-circle orthogonal to $S$. As its convex side is turned toward $A$, we see at once that $A+B+C<2$ right angles.

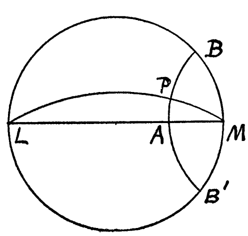

FIG. 2

At the points $A$ of an $H$-straight as $LM$ in Fig. 2 let us erect $H$-perpendiculars meeting $S$ at $B$, $B'$. On each perpendicular lay off a segment $AP$ of constant length in $H$-measure. The locus of these points $P$ is an *equidistant curve.* Such a line in $e$-geometry would be a parallel to the line $LM$. In our model equidistant curves are $e$-circles; the perpendiculars as $ABB'$ are orthogonal to the equidistant curves.

We define a circle and sphere in $H$-space as in $e$-space. Let $A$ be a fixed point; the locus of the points $P$ such that the distance of $P$ from $A$ is a constant $\rho$ in $H$-measure is an $H$-sphere of radius $\rho$ and center $A$. In our model an $H$-sphere is also an $e$-sphere, but its $e$-center is not $A$ except when $A$ coincides with $O$, the center of the fundamental sphere $S$. If $P$ is restricted to an $H$-plane through $A$, the locus is an $H$-circle of radius $\rho$. If we apply the method of inscribed and circumscribed polygons employed in our school geometries we find the length of the circumference of a circle of radius $\rho$ is

$$C = 2\pi R \sinh(\rho/R), \tag{2}$$

while its area in $H$-measure is

$$A = 2\pi R^2(\cosh(\rho/R) - 1).$$

If $R$ is large compared with $\rho$ we have approximately

$$C = 2\pi\rho, \quad A = \pi\rho^2.$$

The volume of an $H$-sphere of radius $\rho$ is

$$V = 4\pi R^3\left(\frac{\rho}{2R} - \frac{1}{4}\sinh\frac{2\rho}{R}\right) \tag{3}$$

$$= \frac{4}{3}\pi\rho^3 \text{ approximately when } \rho/R \text{ is small}.$$

The area of the surface of the sphere is

$$S = 4\pi R^2 \sinh^2(\rho/R).$$

The relations valid in a right triangle whose sides are of lengths $a$, $b$, $c$ in $H$-measure, and whose angles opposite these sides are $A$, $B$, $C$, the latter being a right angle, are

$$\sin A = \frac{\sinh(a/R)}{\sinh(c/R)}, \qquad \tan A = \frac{\tanh(a/R)}{\sinh(b/R)}, \tag{4}$$

$$\cosh(c/R) = \cosh(a/R)\cosh(b/R).$$

Finally let us show how we can find the length of a curve $C$ in this model. If $d\sigma$ is the $e$-length of an element of arc of $C$, its length in $H$-measure is

$$ds = \frac{d\sigma}{1 - \dfrac{r^2}{4R^2}}, \tag{5}$$

where $r$ is the $e$-distance of the element from the center $O$ of $S$.

4. *Elliptic Geometry.* To construct a model of elliptic space in $e$-space we take again an $e$-sphere $S$ of radius $R$ and center $O$. $E$-straights in this model are $e$-circles cutting $S$ in diametral points. $E$-planes are represented in the model by $e$-spheres cutting $S$ along great circles. To diametral points in the model as $A$, $B$, on $S$, correspond but a single point in $E$-space. Thus all $E$-straights are closed curves, two co-planar $E$-straights always cut, and two points determine an $E$-straight. Figures in $E$-space may be moved about freely without distortion. Any straight may be moved to coincide with any other: all have the length $\pi R$.

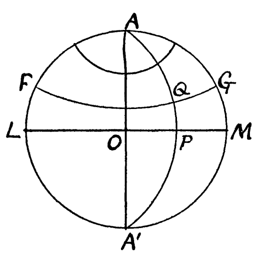

FIG. 3

The model exhibits very clearly that the sum of the angles of a plane triangle $A$, $B$, $C$ is greater than 2 right angles. For if we move the plane $ABC$ so that $A$ coincides with the center of $S$, two of the sides in the model are $e$-straights and the third side $BC$ is an $e$-circle with its concave side turned toward $A$.

All $E$-straights as $PA$ in Fig. 3, cutting $S$ in $A$, perpendicular to a given $E$-straight $LM$ meet in a point $A$ called the pole of $LM$. The length of all these perpendiculars is $\pi R/2$ in $E$-measure. The family of $e$-circles whose centers lie on the straight $AOA'$ perpendicular to $LM$ and which cut the circle $S$ orthogonally are $E$-circles having $A$ as center. Let $FQG$ be one of these cutting $S$ in $F$, $G$. Since $AQ$ is an $E$-radius, the segment $PQ$ is constant along $FG$; hence the curve $FQG$ is a curve whose points lie at the same constant distance from the $E$-straight $LM$. Such curves are called *equidistant curves*; in $e$-geometry they would be a parallel to $LM$.

A peculiarity of $E$-space is made quite clear by our model. Suppose a flat circular disk $ABCD$ in Fig. 4 has its center $K$ at $O$, the center of the fundamental sphere; $AKC$ is a vertical diameter and $BKD$ a horizontal diameter; the points $A\ B\ C\ D$ correspond to 12, 3, 6 and 9 o'clock on the face of a watch. Suppose the disk is moved in a plane so that its center $K$ describes an $E$-straight meeting $S$ in the points $L'$, $L''$. When $K$ is at $L'$, the diameter $AKC$ coincides with $A'L'C'$ on $S$, to which correspond on $S$ the diametral points $A''L''C''$. The right vertical half of the disk $ABC$ is now at $A''B''C''$ and the upper quadrant $AKB$ is now at $A''L''B''$ below the line $L'L''$. Continuing the motion along $L\ O$, we see that when the center of the disk has returned to $O$, the disk coincides with its original position but $A$ and $C$ have interchanged positions. A similar remark holds for a sphere.

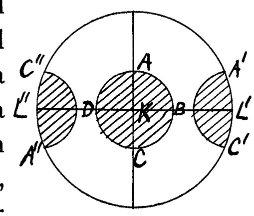

FIG. 4

Another peculiarity of $E$-space is the following. Let $l$ be an $E$-straight, and let $AB$ be two points on either side of $l$. We can pass from $A$ to $B$ without crossing $l$; for let the join of $AB$ cut $S$ in $A'B'$. Then $AA'B'B$ is a continuous path joining $A$, $B$ since $A'=B'$ in $E$-geometry. Thus an $E$-straight does not divide an $E$-plane into two pieces. Similarly an $E$-plane does not divide $E$-space into two pieces. Such is not the case in $e$-space.

Defining a sphere and circle in $E$-space in a manner analogous to $H$-space, we find in a similar manner

$$(6)\qquad c=2\pi R\sin(\rho/R),\qquad A=2\pi R^2(\cos(\rho/R)-1);$$

$$(7)\qquad V=\frac{4}{3}\pi R^3\left(\frac{\rho}{2R}-\frac{1}{4}\sin\frac{2\rho}{R}\right),\ s=4\pi R^2\sin^2(\rho/R).$$

Also for a right triangle we have, analogous to (4),

$$(8)\qquad \sin A=\frac{\sin(a/R)}{\sin(c/R)},\qquad \tan A=\frac{\tan(a/R)}{\sin(b/R)},$$
$$\cos(c/R)=\cos(a/R)\cos(b/R).$$

The $E$-length of an element of arc of $e$-length $d\sigma$ in the model is

$$(9) \qquad ds = \frac{d\sigma}{1 + \frac{r^2}{4R^2}} .$$

We notice (9) goes over into (5) on replacing $R$ by $iR$.

5. *The Parallel Axiom.* As we have observed, Euclid's geometry is characterized by a certain assumption regarding the infinite part of space. In the *Elements* of Euclid it appears as postulate 5 and is known as the *parallel axiom.* From time immemorial this postulate has given offense. Not that anyone doubted its truth; but in the form as given by Euclid it did not seem as intuitively self-evident as the other axioms and postulates. For centuries one believed it was possible to deduce it from the other axioms; and countless have been the attempts to do so. It was reserved to Lobachevsky and Bolyai to show the world that it is possible to deduce a consistent geometry in which the parallel axiom of Euclid does not hold.

The question arises: Is our space euclidean or is it not? To answer this question, Gauss, who was engaged in geodetic work in connection with the great Hannover triangulation, measured the angles of the triangle whose vertices were stations on Hohenhagen, Inselsberg, and the Brocken. As reported by Waltershausen, the sum of these three angles differed from two right angles by about 0.2″. As this amount is quite within the limit of error, these observations leave the question undecided. They do show, however, that if space is not euclidean, the space constant $k$ is small.

Assuming that space is hyperbolic, Lobachevsky gave a method of estimating a lower limit of $k$ as follows. Let $AB$ in Fig. 5 be opposite extremities of the earth's orbit. Let $S$, the sun, be midway between. Let a star $D$ lie on the perpendicular $SD$. Let $AD$ and $BD$ be $H$-straights whose $e$-tangents at $A$ and $B$ are $ACA'$, $BCB'$ meeting at a point $C$ on the perpendicular $SD$.

If space were euclidean an observer would see the star at $C$, and in passing from $A$ to $B$ the star would appear to move through the angle $A'C'B'$. Astronomers call half this angle the parallax $p$ of the star. Then in the $e$-right triangle $ACS$,

$$p=C=90-A.$$

The distance of the star in $e$-space i.e. as estimated by astronomers is

$$SC=AS\cdot\tan A=AS\cdot\text{ctn }p.$$

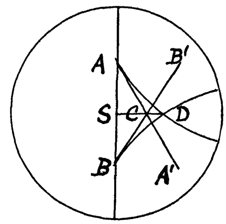

FIG. 5

If space were hyperbolic, we would have to consider the $H$-right triangle $ASD$. Let $AS=a$, $SD=\rho$ in $H$-measures. Then, by (4), we have

$$(10)\quad \tanh(\rho/R)=\sinh(a/R)\tan A=\frac{\sinh(a/R)}{\tan p}=\frac{a}{Rp},$$

approximately, since $a/R$ is small. Hence

$$\sinh(\rho/R)=\frac{a}{\sqrt{p^2R^2-a^2}}.$$

This requires that

$$p^2R^2>a^2, \qquad \text{or} \qquad R>a/p.$$

As there are stars whose parallax is less than $0.05''$ this gives

$$R>4\cdot 10^6\cdot a.$$

Let us suppose now that space were elliptic. Suppose a star $A$ were moving along an $E$-straight, as $LAM$ meeting $S$ at $L$, $M$. An observer at $O$ sees the star in the direction $OA$. As $E$-straights are closed lines, he also sees it in the opposite direction $OLMA$. Thus to stars having no proper motion or small proper motion, should correspond antistars.

We do not observe such antistars, and we may therefore suppose that $R$ is so large that light is absorbed to such an extent that the antistars cannot be observed. Let us apply this above to the sun, whose magnitude is about $-26.5$.

Assuming we could detect an anti-sun of magnitude 15 let us use the relation

$$m = -5 \log p,$$

where $m$ is the magnitude of the sun at a distance corresponding to a parallax $p$. Here $m = 15$, hence $p = 0.001''$.

To this parallax corresponds a distance $a' = 2 \cdot 10^8 \cdot a$, where $a = OA$, the distance of the earth to the sun. As $a + a' = \pi R$, we have

$$R = \frac{2a}{\pi} \cdot 10^8 = 6 \cdot 10^7 \cdot a \text{ , approximately.}$$

The distribution of the stars in space has excited the interest of philosophers from very early days; scientifically the question was first taken up by Herschell (1784). The question arises: Is all space loosely filled with stars or do they cease to exist outside a certain sphere? Already in 1826 Olbers noticed that in the former case the whole sky would be as bright as the sun, unless the light is partly absorbed in its passage through space. C. Neumann and v. Seelinger (1896) have noted another difficulty. If we assume that Newton's gravitation law of inverse squares holds good throughout all space, we must suppose the density of cosmic matter is zero for an infinite space. The milky way or galaxy to which our sun belongs is estimated to be within a sphere of radius $3 \cdot 10^4$ light years. Beyond this are the star clusters and spiral nebulae; the nearest of which is perhaps $10^6$ light years distant. One light year $= 6 \cdot 10^4$ orbrads, or astronomical units.

6. *Analytic Formulation.* In the foregoing we have been able to present many of the results of non-euclidean geometry in a way easy to visualize. In order to acquaint you with some of the more recent work in this field it is necessary to have recourse to analysis. It rests on the 1868 paper of Riemann mentioned above.

Geometers define a point in space in a great variety of ways by means of three coordinates. Without specifying what coordinates we employ we will denote them by $x_1$, $x_2$, $x_3$.

We allow these $x$'s to range continuously over a certain set of values; to each set of values in this range corresponds a point which we denote by $x=(x_1x_2x_3)$. We call the point $x+dx$ a nearby point, and say the distance of $x+dx$ from $x$ is $ds$, where

$$ds^2=\sum a_{ij}dx_idx_j\ ; \qquad a_{ij}=a_{ji}\ ; \qquad (i,j=1,2,3)\ . \tag{11}$$

Let $x_1=\varphi_1(t)$, $x_2=\varphi_2(t)$, $x_3=\varphi_3(t)$; when $t$ ranges over a certain interval $(\alpha,\beta)$ we say $x$ describes an arc of a curve $C$ whose length is

$$l=\int_\alpha^\beta \frac{ds}{dt}dt\ . \tag{12}$$

If

$$\delta\int ds=0\ , \tag{13}$$

we say $C$ is a geodesic. In $e$-geometry these are straights.

Let $C$, $C'$ be two curves meeting in a point $x$; the angle $\theta$ between these curves is defined by the equation

$$\cos\theta=\sum a_{ij}\frac{dx_i}{ds}\cdot\frac{dx_j'}{ds'}\ . \tag{14}$$

We call

$$\xi_i=\frac{dx_i}{ds}\ , \qquad (i=1,2,3)\ , \tag{15}$$

the *direction constants* at the point $x$ along $C$. A pencil of geodesics whose direction constants are

$$\gamma\xi_i+\mu\xi_i' \tag{16}$$

define a *geodesic surface*. In $e$-geometry this is a plane. We call

$$a=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} \tag{17}$$

the *determinant of the form* (11) and define the *element of volume* by

$$dv=\sqrt{|a|}\cdot dx_1dx_2dx_3\ . \tag{18}$$

These definitions show that the fundamental elements of a given geometry depend on the quadratic differential form (11). We say this form defines the metric of the particular geometry or space considered. Let us consider two geometries defined by the two forms (11) and

$$ds'^2 = \sum a'_{ij} dx'_i dx'_j \; ;$$

we ask when are these geometries essentially the same. This occurs when there is a continuous one to one correspondence between the $x$, $x'$ such that on changing from one set of variables to the other $ds$ goes over into $ds'$ and conversely. In fact, to a figure $F$ in the $x$-space will correspond a figure $F'$ in the $x'$-space such that corresponding lengths and angles in the two figures have the same values. In this case we say the two forms $ds$ and $ds'$ are *equivalent*, so that to equivalent forms correspond the same geometry, at least for not too large regions.

For example, consider the geometry on a cylinder or cone. By cutting the surface along a rectilinear generator it may be rolled flat on a plane without distortion. Geodesics on one of these surfaces become $e$-straights on the plane. The geometry on these surfaces is thus essentially the same as $e$-plane geometry for figures not too large.

To find analytic conditions in order that $ds' = ds$, we introduce the following symbols due to Christoffel:

$$(19) \qquad \Gamma_{\alpha\beta,\lambda} = \begin{bmatrix} \alpha\beta \\ \lambda \end{bmatrix} = \frac{1}{2}\left(\frac{\partial a_{\alpha\lambda}}{\partial x_\beta} + \frac{\partial a_{\beta\lambda}}{\partial x_\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x_\lambda}\right),$$

$$(20) \qquad \Gamma^{\lambda}_{\alpha\beta} = \begin{Bmatrix} \alpha\beta \\ \lambda \end{Bmatrix} = \sum_i a^{i\lambda} \begin{bmatrix} \alpha\beta \\ i \end{bmatrix},$$

$$(21) \qquad \frac{\partial a_{\lambda\mu}}{\partial x_\alpha} = \Gamma_{\lambda\alpha,\mu} + \Gamma_{\mu\alpha,\lambda} \, .$$

Here $a^{\lambda\mu}$ = minor of $a_{\lambda\mu}$ divided by $a$.

We need also a four-index symbol due to Riemann:

$$(22)\qquad (\mu\lambda, jk) = R_{\lambda\mu,jk} = \frac{\partial \Gamma_{\lambda j,\mu}}{\partial x_k} - \frac{\partial \Gamma_{\lambda k,\mu}}{\partial x_j} + \sum_{i\alpha} a^{i\alpha}\left(\Gamma_{\lambda k,\alpha}\Gamma_{\mu j,i} - \Gamma_{\lambda j,\alpha}\Gamma_{\mu k,i}\right),$$

from which we form

$$(23)\qquad \{\lambda\alpha, jk\} = R^{\alpha}_{\lambda jk} = \sum_{\mu} a^{\alpha\mu}(\mu\lambda, jk)$$

$$= \frac{\partial \Gamma^{\alpha}_{\lambda i}}{\partial x_k} - \frac{\partial \Gamma^{\alpha}_{\lambda k}}{\partial x_j} + \sum_i \left(\Gamma^{\alpha}_{ik}\Gamma^{i}_{\lambda j} - \Gamma^{\alpha}_{ij}\Gamma^{i}_{\lambda k}\right);$$

$$(24)\qquad R_{\lambda k} = \sum_{j\mu} a^{j\mu} R_{\lambda\mu,jk} = \sum_i R^{i}_{\lambda ik},$$

$$(25)\qquad R = \sum_{\lambda k} a^{\lambda k} R_{\lambda k}.$$

We can now state a necessary condition in order that $ds'^2$ be equivalent to $ds^2$, viz.:

$$(26)\qquad (\alpha\delta, \beta\gamma)' = \sum_{rkih} (rk, ih)\frac{\partial x_r}{\partial x'_\alpha}\,\frac{\partial x_i}{\partial x'_\beta}\,\frac{\partial x_h}{\partial x'_\gamma}\,\frac{\partial x_k}{\partial x'_\delta}$$

where the symbol on the left refers to $ds'$. This is called the condition of integrability. There is an important case when this relation is identically satisfied, viz., when the coefficients $a_{ij}$ in $ds^2$ satisfy the relations

$$(27)\qquad (rk, ih) = K(a_{ri}a_{kh} - a_{rh}a_{ki})$$

where $K$ is a constant.

As an example let us consider a surface $S$ whose coordinates are expressed as functions of two parameters $x_1$, $x_2$. Let the metric on $S$ be given by

$$ds^2 = a_{11}dx_1^2 + 2a_{12}dx_1dx_2 + a_{22}dx_2^2 .$$

The 16 symbols $(\alpha\beta, \gamma\delta)$ are here all 0 except

$$(12,12) = -(12,21) = -(21,12) = (21,21).$$

This symbol (12, 12) is closely related to what Gauss called the curvature of a surface at a given point. In fact let $\nu$ be the normal to $S$ at a point $x$. Any plane through $\nu$ cuts out a normal section $C$ which will have a certain curvature at $x$. There are two planes at right angles to each other for which the curvatures $\kappa_1$, $\kappa_2$ are a maximum and a minimum. Gauss calls

$$k=\kappa_1 \cdot \kappa_2 \tag{28}$$

the curvature of $S$ at $x$. We find now that

$$k=\frac{(12,12)}{a}\,. \tag{29}$$

In general $k$ is not a constant as $x$ ranges over $S$. If we set $r=1$, $k=2$, $i=1$, $h=2$ in (27) we see $K=k$ at $x$. For this reason a space whose metric (11) satisfies (27) is said to be of constant curvature $K$.

We have now the following facts:

1) If two metrics have the same constant curvature they are equivalent.

2) In space of constant curvature there are $\infty^6$ displacements or transformations which leave $ds$ unaltered. A trieder whose vertex is $A$ can be displaced so as to coincide with a given trieder whose vertex is $A'$.

3) In space of constant curvature $k$, we may choose as coordinates $x_1$, $x_2$, $x_3$ such that $ds^2$ takes the canonical form

$$ds^2=\frac{dx_1^2+dx_2^2+dx_3^2}{\left\{1+\frac{k^2}{4}(x_1^2+x_2^2+x_3^2)\right\}^2}\,. \tag{30}$$

When $k^2<0$ we get the geometry of Lobachevsky and Bolyai; when $k^2>0$ we get elliptic geometry and another geometry in which two straights cut in two points. For $n=2$ this gives geometry on a sphere. We may call this second type of geometry for a positive $k$, *spherical* or *sphero-elliptic*. When $k=0$, the metric is euclidean.

4) The necessary and sufficient condition that $ds^2$ can be brought to the form $ds^2 = c_1 dx_1^2 + c_2 dx_2^2 + c_3 dx_3^2$, where the $c$'s are constants, is that the symbols $(\alpha\beta, \gamma\delta)$ should all be zero. In case the $c$'s are all positive we can reduce this to

$$ds^2 = dy_1^2 + dy_2^2 + dy_3^2$$

i.e. this space is euclidean.

We return now to the general case. Suppose the space $S$ which we are considering is not of constant curvature, what then? We will suppose its metric is given by (11). Let $x$ be a point of $S$ and $\alpha$ an arbitrary plane through $x$. The geodesics through $x$ whose tangents lie in $\alpha$ form a geodesic surface $G_\alpha$ whose equations in parametric form are, say,

$$x_1 = x_1(u_1, u_2) \ , \qquad x_2 = x_2(u_1, u_2) \ , \qquad x_3 = x_3(u_1, u_2) \ .$$

From them we get

$$dx_\lambda = \frac{\partial x_\lambda}{\partial u_1} du_1 + \frac{\partial x_\lambda}{\partial u_2} du_2 \ ,$$

which in (11) gives the metric on $G_\alpha$ viz.:

(31) $$d\sigma^2 = \alpha_{11} du_1^2 + 2\alpha_{12} du_1 du_2 + \alpha_{22} du_2^2.$$

We define now the curvature of $G_\alpha$ at $x$ by

(32) $$k_\alpha = \frac{(12, 12)_\alpha}{\alpha} \ ,$$

where $\alpha$ is the determinant of (31) and $(12, 12)_\alpha$ is the Riemann symbol relative to the metric (31). If we let the plane $\alpha$ turn about $x$ we find there are three positions orthogonal to each other corresponding to maximum and minimum values of $k$; call these $k_1$, $k_2$, $k_3$. If now $\xi_1$, $\xi_2$, $\xi_3$ of (15) are the direction constants of the normal to $\alpha$ we have

(33) $$k_\alpha = k_1 \xi_1^2 + k_2 \xi_2^2 + k_3 \xi_3^2$$

as the curvature of space at $x$ for a given orientation $\alpha$.

Levi-Civita has introduced a notion which has led to important generalizations of our ideas of space, in the hands of

Weyl and others. Let $C$ be a curve on the surface $S$; $x$ and $x+dx$ are two nearby points, $\alpha$ is an infinitesimal vector lying in the tangent plane $\tau$ at $x$. Let us move $\alpha$ from $x$ to $x+dx$ parallel to itself. This vector $\beta$ will not in general lie in $S$. Let us therefore resolve $\beta$ into two components, one lying in the tangent plane $\tau'$ at $x+dx$, call it $\alpha'$, and the other component normal thereto.

If $\xi_1, \xi_2, \xi_3$ are the direction cosines of $\alpha$ and

$$\xi_i = \xi_i + \frac{d\xi_i}{ds} ds$$

those of $\alpha'$, we have

$$\frac{d\xi_i}{ds} + \sum_{jk} \xi_j \frac{dx_k}{ds} \left\{ \begin{matrix} jk \\ i \end{matrix} \right\} = 0 .$$

We generalize the foregoing considerations as follows. Let an elementary vector in space, whose components are $A', A'', A'''$ be displaced from a point $x$ to $x+dx$ such that the new components $A^i + dA^i$ satisfy

$$dA^i + \sum_{jk} A^j dx_k \left\{ \begin{matrix} jk \\ i \end{matrix} \right\} = 0 , \qquad (i=1,2,3) . \tag{34}$$

Such a displacement is called an *infinitesimal parallel displacement*, or a *geodetic displacement.*

If $dx_1, dx_2, dx_3$ are the components of the infinitesimal vector along the tangent to a geodesic $G$ in space, we have

$$A^i = \frac{dx_i}{ds}, \qquad (i=1,2,3);$$

substituting this in (34), we find

$$\frac{d^2 x_i}{ds^2} + \sum_{jk} \frac{dx_j}{ds} \frac{dx_k}{ds} \left\{ \begin{matrix} jk \\ i \end{matrix} \right\} = 0 , \qquad (i=1,2,3) , \tag{35}$$

which are the equations of $G$.

Hence the displacement of a tangent along a geodesic is a case of infinitesimal parallel or geodetic displacement. Let us displace geodetically the elementary vector $(A', A'', A''')$

around a closed curve $C$. We find that the components $A^i$ have changed by the amount

$$\Delta A^i = -\int_C \sum_{\lambda jk} A^\lambda R^i_{\lambda jk} dx_j dx_k \,, \qquad (i=1,2,3)\,.$$

If the curvature tensor is 0, $\Delta A^i$ is zero and conversely. Thus only in flat spaces does a vector remain unchanged when displaced geodetically along a closed curve. For example, if we displace geodetically a vector over the sides of a geodesic triangle on a sphere we find its direction has changed at the end of the circuit by an amount equal to the spherical excess of the triangle.

On the other hand, if the vector $A^\mu$ whose squared length is $l^2$ is moved parallel to itself around a small circuit we find

$$d \cdot l^2 = \sum_{\mu\nu\sigma} 2K_{\mu\nu,\sigma} A^\mu A^\nu dx_\sigma \,,$$

where

$$K_{\mu\nu,\sigma} = \frac{1}{2}\left(\frac{\partial a_{\mu\nu}}{\partial x_\sigma} - \Gamma_{\sigma\mu,\nu} - \Gamma_{\sigma\nu,\mu}\right). \qquad (36)$$

As here $K_{\mu\nu,\sigma}=0$ by (21) we see that $d \cdot l^2 = 0$, i.e., $l$ is unchanged.

Let $dx = PA$, $\delta x = PB$ be two elementary vectors; they determine a geodetic surface $G$. On displacing geodetically $PA$ along $\delta x$ let $A$ pass to $C$; we find if we displace $PB$ geodetically along $dx$, that $B$ coincides with $C$. Let us move an elementary vector $v$ geodetically around the parallelogram $PACB$ whose area we call $\Delta\alpha$. At the end of the circuit we will suppose $v$ has become $v'$, making an angle $\Delta\theta$ with $v$. We find now that the curvature $k$ of $G$ at $P$ is

$$k = -\frac{\Delta\theta}{\Delta\alpha}\,, \qquad (37)$$

which gives an elegant interpretation of $k$.

7. *Higher Dimensional Space.* According to Minkowski (1908) our universe is a four-dimensional manifold; we must therefore say a few words about four-dimensional or more

generally $n$-dimensional space. To "the man in the street" a 4-way space is a thing of mystery and awe. He has heard that if space had a fourth dimension a person could escape from a room, closed as to three-way space; a closed surface could be turned inside out like a glove; knots in closed strings could be untied etc. Such facts were used by spiritualists to make plausible the manifestations of the so called spirit world; as, for example, by F. Zöllner, professor of astrophysics at the University of Leipzig.

This is not the point of view of the mathematician. To him $n$-way space is a figment of the brain. Let $x_1, \cdots, x_n$ be $n$ variables; the complex $(x_1, \cdots, x_n)$ he regards as defining a point $x$, and $n$-way space is the totality of these points. Let the parameter $u$ range over a certain set of values, say $\alpha \leqq u \leqq \beta$; the points $x$ whose coordinates are

$$x_1 = x_1(u), \quad \cdots, x_n = x_n(u)$$

are called a curve, or a segment of a curve. Let $v$ be another parameter; the points whose coordinates are

$$x_1 = x_1(u, v), \quad \cdots, x_n = x_n(u, v)$$

lie on a surface, etc. A linear relation between the $x$'s, as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + c = 0,$$

defines a plane; the equation

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2 = R^2$$

defines a sphere whose center is $a$ and whose radius is $R$, etc.

One sees that the method of procedure is simply one of generalization of the equations of ordinary analytic geometry. If one wishes an $n$-dimensional non-euclidean space, we replace the metric $ds^2 = dx_1^2 + \cdots + dx_n^2$ by

$$ds^2 = \sum_{i,j} a_{ij} dx_i dx_j, \quad a_{ij} = a_{ji}, \quad (i, j = 1, 2, \cdots, n). \tag{38}$$

The only difference between this and (11) is the number of variables $x$. All the notions of 3-dimensional non-euclidean space developed in the previous section may be extended at once to $n$-way space.

I suppose the layman will think $n$-dimensional geometry is a crazy fiction, and mathematicians who study it rather wanting in common sense; yet such is not the case. In the first place I would observe that all abstract geometries are products of our mind, the euclidean geometry of our text books being very closely related to the geometry of physical space as manifested to us by our sense perceptions. There is therefore no a priori reason why we should not free ourselves from the trammels of three dimensions and see what is to be gained by imagining space of higher dimensions.

The first to suggest a space of more than three dimensions as far as I have ascertained was Lagrange. In his *Théorie des Fonctions* (1797) he says; "Thus we may regard mechanics as a geometry of four dimensions, and analytic mechanics as an extension of analytic geometry."

Cayley in the early 40's saw clearly the service that space of higher dimensions might render in studying ordinary geometry and Grassmann in 1844 treated it systematically in his Ausdehnungslehre. Gradually it has become an integral part of geometry. Even the pre-Einstein physicist found it convenient, so for example in the kinetic theory of gases. The motion of each molecule of the gas is given by 6 coordinates, viz.: the three coordinates $x,y,z$ of its center of mass and the three components $u,v,w$ of its velocity. Consider a gas formed of $N$ molecules in a closed container; it is convenient to represent the state or phase of the gas by a point in 6 $N$-way space. As there are something like $3 \cdot 10^{19}$ molecules in 1 ccm at $0^0$C, and pressure of 1 atmosphere, one sees that the dimensionality of this space is a tidy little number.

We need one more notion before taking up the ideas of Einstein. When Kepler and Newton came to study from a fresh point of view the motion of the planets, they found ready for use the properties of conic sections developed by the Greek geometers centuries before. With no thought of gain, actuated only by an ideal love of science, Appolonius of Perga and many other ancient geometers had studied the conic sections and we may well believe that the whole history of modern astronomy would have been different if this theory had not been known.

In a manner quite similar, Einstein found ready for use a calculus without which there would have been no general theory of relativity. This calculus was called by its inventors, Ricci and Levi-Civita, the absolute differential calculus; the name tensor analysis seems to be preferred to-day.

What is a tensor? Consider the $n$ quantities

$$A' = dx_1, \cdots, A^{(n)} = dx_n; \tag{39}$$

they are the components of an infinitesmal vector $A$. Let us introduce $n$ new variables $x_1', \cdots, x_n'$ and set

$$p_{ij} = \frac{\partial x_i}{\partial x_j'}, \qquad p_{ij}' = \frac{\partial x_i'}{\partial x_j}.$$

Then the $n$ quantities

$$\bar{A}' = dx_1', \cdots, \bar{A}^{(n)} = dx_n'$$

are related to the original $n$ quantities (39) by the equations

$$\bar{A}^{(i)} = \sum_j p_{ij}' A^{(j)}, \qquad (i, j = 1, 2, \cdots, n). \tag{40}$$

In general any $n$ quantities $A', A'', \cdots, A^{(n)}$ which are transformed, on introducing $n$ new variables, according to (40) form a contravariant tensor of order 1.

Similary, the $n^2$ quantities $a^{ij}$ relative to (38) go over, on changing from the $x$ to the $x'$ variables, into

$$\bar{a}^{ij} = \sum p_{i\lambda}' p_{j\mu}' a^{\lambda\mu}.$$

Any $n^2$ quantities $A^{ij}$ which are transformed in this manner form a contravariant tensor of order 2, and so on.

Let $\varphi(x_1, x_2, \cdots, x_n)$ have continuous first derivatives

$$A_1 = \frac{\partial \varphi}{\partial x_1}, \cdots, A_n = \frac{\partial \varphi}{\partial x_n}.$$

On changing to the $x'$ variables, these become

$$\bar{A}_i = \frac{\partial \varphi}{\partial x_i'} = \sum_j p_{ij} A_j.$$

Any $n$ quantities $A_1, \cdots, A_n$ which are transformed in a similar manner form a covariant tensor of order 1.

A set of $n^2$ quantities $A_{ij}$ which are transformed by

$$\bar{A}_{ij} = \sum_{\lambda\mu} p_{i\lambda} p_{j\mu} A_{\lambda\mu} \tag{41}$$

form a covariant tensor of order 2. The coefficients $a_{ij}$ in (38) form a tensor of order two. The $n^4$ Riemannian symbols $R_{\lambda\mu,jk} = (\mu\lambda, jk)$ formed on the general metric (38) transform as in (26); they form thus a covariant tensor of order 4.

If the law of transformation of $n^m$ quantities involves $\alpha$ factors $p_{ij}$ and $\beta$ factors $p'_{ij}$ where $m = \alpha + \beta$ we say they form a tensor of order $m$, covariant of order $\alpha$ and contravariant of order $\beta$. Thus the $R^{\alpha}_{\lambda jk}$ of (23) form a tensor of order 4, covariant of order 3, contravariant of order 1. The $R_{\lambda k}$ of (24) form a covariant tensor of order 2. The $R$ of (25) is altogether unchanged by transformation; it is an invariant; so is $ds^2$ in (38); so is $\cos\theta$ as defined in (14) relative to the $ds^2$ in (38).

8. *Restricted Relativity*. Let us return for a moment to the fundamental concepts of geometry, the straight line, the plane etc. All our mechanics, physics etc. were founded on euclidean geometry until the advent of Einstein. For example all pre-einsteinean optics rests on the assumption that light travels in a euclidean straight line and astronomers use this theory to make delicate tests in constructing plane mirrors. Now eclipse observations seem to show that light in passing a massive body like the sun is deflected; its path apparently resembles a very flat hyperbola. If space is not euclidean, what kind of a space is it?

Almost 50 years ago (1870) Clifford inspired by Riemann's great paper of 1868 held the following beliefs:

1) That small portions of space are, in fact, of a nature analogous to little hills on a surface which is on the average flat; namely that the ordinary laws of geometry are not valid in them.

2) That this property of being curved or distorted is continually being passed on from one portion of space to another after the manner of a wave.

3) That this variation of the curvature of space is what really happens in that phenonenon which we call the motion of matter, whether ponderable or ethereal.

4) That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity.

Clifford never succeeded in putting these ideas on a solid footing; they were too far in advance of the epoch. Instead of being realized by one grand effort of genius the fates that rule the evolution of human thought decreed that they should be unfolded to our eyes in a series of lesser steps; non-euclidean differential geometry, tensor analysis, the Michelson-Morley experiment, the laborious efforts of Lorentz, M. Abraham, Larmor, Poincaré, leading up to the restricted relativity theory of Einstein (1905). This theory welded into a physical entity the 3 coordinates of space $x,y,z$ and the time $t$. Minkowski began his Cologne address (1908) with these revolutionary words:

"The views of space and time which I wish to lay before you have grown up from the fields of experimental physics. In this fact lies their strength. Their tendency is radical. From this hour onward, time and space as independent elements must sink down to shadows, and only a union of both shall preserve an independent existence."

From this standpoint, only *events* exist, a happening at a given place $x,y,z$ and time $t$. To specify an event, these 4 numbers are always necessary; the universe is a 4-dimensional continuum whose metric is given by

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 . \tag{42}$$

As we have seen this is a space of constant curvature 0; and any section of it corresponding to $t$ = constant is euclidean.

The restricted relativity theory made a great stir in the scientific world; its teachings ran sheer counter to notions which since the days of Newton had formed the ground work of our scientific thinking. Newton began his *Principia* by postulating an absolute space and an absolute time; relativity relegated these to the scrap heap. Simultaneity of two events lost its former absolute meaning and was shown to be a relative concept depending on the motion of the observer; in a similar manner the mass of a body was no longer an absolute attribute. The luminiferous ether which Lord Kelvin in his Baltimore Lectures (1888) said we knew more about than we did of air, water, glass, iron was declared to be non existent.

No such revolutionary ideas had shaken the scientific world since Copernicus removed the center of our little world from the earth to the sun.

In the simplest manner it extricated physicists from a most embarassing position, bringing unity and clarity where before confusion and obscurity had reigned. It required us indeed to give up certain ways of thinking which long custom had led us to accept as necessary forms of thought; but so did the teaching of Copernicus, Bruno, Kepler, and Galileo.

9. *General Relativity.* Einstein's theory of 1905 was tied down by two restrictions; only rectilinear uniform motions were considered and the velocity of light was assumed to be strictly constant. In 1914 he proposed a broader theory in which both these restrictions were removed. In this theory the metric (42) was replaced by

$$ds^2 = \sum a_{ij} dx_i dx_j \,, \qquad a_{ij} = a_{ji} \,, \quad (i,j = 1,2,3,4). \tag{43}$$

Here one of the variables, as $x_4$, is the time coordinate, and the $a$'s are functions of the $x_1, \cdots, x_4$ such that in the vicinity of a given point $x$ the form (43) reduces to (42) for a properly chosen set of coordinates. All equations describing physical phenomena were to have tensor form. For example, let $A_{\lambda\mu}$ be the 16 components of a tensor and suppose the 16 equations

$$A_{\lambda\mu} = 0 \,, \qquad (\lambda, \mu = 1,2,3,4) \tag{44}$$

expressed a physical fact. On changing to another set of coordinates $x_1', x_2', x_3', x_4'$ the components $A$ become by (41)

$$\bar{A}_{ij} = \sum_{\lambda\mu} p_{i\lambda} p_{j\mu} A_{\lambda\mu} \,, \qquad (i,j = 1,2,3,4) \,.$$

Here each term $A_{\lambda\mu}=0$; the equations (44) are replaced by the equations

$$\bar{A}_{ij}=0\ , \qquad (i,j=1,2,3,4)\ ,$$

which bear the same relation to the transformed $ds'^2$ as $A_{ij}$ to (43). We express this condition by saying that the equations of mathematical physics must be invariant relative to all transformations of the coordinates.

In general, the metric (43) is not euclidean; the path of a free particle will not be a euclidean straight; its deviation from such a path is due to gravitational masses. Einstein assumes the warp or twist of space, characterized by a purely geometrical tensor, is measured by a purely physical tensor. As geometrical tensor, Einstein takes (24), (25)

$$G_{\lambda\mu}=T_{\lambda\mu}-\tfrac{1}{2}a_{\lambda\mu}P\ ;$$

as physical tensor he takes for a continuous medium of proper density $\rho$

$$T_{\lambda\mu}=\sum_{ij}\rho a_{\lambda i}a_{\mu j}\frac{dx_i}{ds}\frac{dx_j}{ds}\ .$$

Then the coefficients $a_{ij}$ of the metric (43) are determined by

$$(45) \qquad G_{\lambda\mu}=-\kappa T_{\lambda\mu}\ ;$$

where $\kappa$ is a universal constant. Where there is no matter, $\rho=0$ and $T_{\lambda\mu}=0$; then the 10 unknown coefficients $a_{ij}$ determining the metric are given as solutions of the ten partial differential equations

$$(46) \qquad G_{\lambda\mu}=0\ .$$

As a special case consider the warp of space produced by a single heavy body as the sun. Einstein finds

$$(47) \qquad ds^2=-\frac{dr^2}{1-\mu/r}-r^2d\varphi^2-r^2\cos^2\varphi d\theta^2+\frac{dx_4^2}{1-\mu/r}\ .$$

Here $r,\varphi,\theta$ are polar coordinates, $x_4$ is the time coordinate and $\mu=2km/c=3\cdot 10^5$ c.g.s. units, $m=$ mass of sun, $c=3\cdot 10^{10}$ cm.

per sec. = velocity of light. For $x_4$ = const., $dx_4 = 0$, and (47) gives as metric of the 3-way space around the sun

$$(48) \qquad -ds^2 = \frac{dr^2}{1-\mu/r} + r^2 d\varphi^2 + \cos^2\varphi dA^2 .$$

It is not euclidean. The path of a particle freely moving about the sun is a geodesic. Such a particle in Newton's theory describes an ellipse; in Einstein's theory it describes a slowly turning ellipse. This is verified in the case of Mercury. Einstein's theory also requires that light shall be bent when passing a heavy body and the spectral lines shall shift toward the red. This has also been verified.

In 1917 Einstein published his *Cosmological considerations*. As we do not know much as to the distribution of cosmic matter in the depths of space, such considerations are highly hypothetical, but it is interesting to see what results one can deduce. A number of reasons led Einstein to adopt as metric of the time-space universe

$$(49) \qquad ds^2 = c^2 dt^2 - \frac{dx_1^2 + dx_2^2 + dx_3^2}{\left[1 + \frac{k^2}{4}(x_1^2 + x_2^2 + x_3^2)\right]^2} .$$

For $dt = 0$ this gives the metric of elliptic or sphero-elliptic geometry. Among the reasons which influenced this choice we may note the following:

1) If space were infinite the values of the $a_{ij}$ at infinite distance would be troublesome.

2) Identifying cosmic matter with gas molecules in isothermic equilibrium we may use Bolzmann's formula

$$\rho_0/\rho = e^{(v-v_0)/RT} ,$$

where $\rho$ is density and $v$ velocity. If now $\rho$ approaches zero while $\rho_0$ is finite, $v$ must become infinite; that is, the stars would have very large velocities. On the other hand $\rho$ must approach zero unless the force of attraction of the matter of the universe, supposed infinite, on a given particle is indeterminate or infinite.

The coefficients $a_{ij}$ of the cosmological metric (49) do not satisfy the field equations (45), and Einstein was compelled to introduce a new universal constant $\lambda$. The equations (45) are replaced by

$$(50) \qquad G_{\alpha\beta} - \lambda a_{\alpha\beta} = -\kappa T_{\alpha\beta} \; ; \qquad (\alpha,\beta = 1,2,3,4) \; .$$

The constants $\kappa,\lambda$ are related by

$$(51) \qquad 2\lambda = \kappa\rho \; , \qquad \kappa\rho = \frac{2}{R^2} \; ,$$

where $1/R^2 = k$ is the curvature of the $x_1,x_2,x_3$ space, and $\rho$ is the density of cosmic matter supposed on the whole to be constant. Now Kapteyn estimates that there are 80 suns of about the same mass as ours in a cube about our sun having sides of lengths 10 parsecs or $3 \cdot 10^{19}$ cm. Then

$$\rho = 5 \cdot 9 \cdot 10^{-24} \, .$$

Hence

$$R = 1 \cdot 3 \cdot 10^{25}$$

$$= 9 \cdot 10^{11} \text{ orbrads} \; .$$

DeSitter has given (1917) another solution of the field equations (50). He finds that the $a_{ij}$ of the metric

$$(52) \quad ds^2 = -R^2 \{ d\rho^2 + \sin^2\rho (d\psi^2 + \sin^2\psi d\theta^2) \} + c^2 \cos^2\rho dt^2$$

satisfy (50) if we take

$$(53) \qquad \rho = 0 \; , \qquad \lambda = \frac{3}{R^2} \; .$$

This metric is spherical with respect to the time coordinate $t$, as well as to the space coordinates. According to this theory we should find:

1) That light changes its wave length $\lambda$, due to *mere* distance according to the formula

$$(54) \qquad \frac{\Delta\lambda}{\lambda} = \pm \tan \rho \; ,$$

i.e., the spectra of distant stars should show a systematic shift toward the red.

2) Celestial bodies at a great distance have on the average a greater velocity than those near to us. Quite recently astronomers have succeeded in measuring the apparent motion of distant nebulae and star clusters, and both of these predictions have been verified thus far.

L. Silberstein (1924) has used the DeSitter shift toward the red to estimate the space constant $R$. He finds approximately

$$R=6.10^{12} \text{ astronomical units,}$$

a result which agrees well with the estimate made by Einstein, as above mentioned.

10. *Electro-Magnetic Phenomena.* In our sketch of Einstein's theory thus far, we have considered only the effect of gravitation; the equations (50) which determine the $a_{ij}$ of the 4-dimensional universe take no account of electromagnetic phenomena. To account for this side of nature, Einstein introduces a covariant tensor $(\varphi_1, \cdots, \varphi_4)$ which he identifies with the electromagnetic potential. By means of the $\varphi$'s and the $g$'s it is easy to write down the equations corresponding to the field equations of Maxwell.

According to this theory the metric of the universe is determined by gravitational matter and not at all by electric masses which may be present; it seems however as though electric masses should play some part in determining the metric of the universe.

In 1918 H. Weyl, generalizing the parallel displacement of Levi-Civita, showed how a 4-dimensional geometry could be constructed, embracing gravitation and electricity in an organic manner. Weyl's ideas have been further extended by Eddington (1921), by Einstein himself (1923–25), by Wirtinger, Cartan, Schouten, Eisenhart, Veblen, and many others. In fact a new branch of differential geometry has been created in the last few years which includes the geometry of Riemann as a special case. By identifying certain tensors which arise in this theory with gravitational and electrical phenomena we obtain various ways of defining the structure of physical space.

In Riemann's geometry we start with a given metric (11) or (38) by means of which we form the Christoffel symbols $\left\{ {\lambda\mu \atop \alpha} \right\}$. Then with the vector $A^\alpha$ at $x$ we associate the vector $A^\alpha + \delta A^\alpha$ at the point $x+dx$ where

$$\delta A^\alpha = dA^\alpha + \sum_{\lambda\mu} A^\lambda \left\{ {\lambda\mu \atop \alpha} \right\} dx_\mu . \tag{55}$$

Similarly with the vector $A_\alpha$ we associate the vector $A_\alpha + \delta A_\alpha$ at the point $x+dx$ where

$$\delta A_\alpha = dA_\alpha - \sum_{\lambda\mu} A_\lambda \left\{ {\mu\alpha \atop \lambda} \right\} dx_\mu . \tag{56}$$

The equations (55) and (56) define covariant differentiation in Riemannian geometry. In the new geometry we define these differentials by analogous expressions, viz.:

$$\delta A^\alpha = dA^\alpha + \sum_{\lambda\mu} A^\lambda \Gamma^\alpha_{\lambda\mu} dx_\mu , \tag{57}$$

$$\delta A_\alpha = dA_\alpha + \sum_{\lambda\mu} A_\lambda \Delta^\alpha_{\lambda\mu} dx_\mu \tag{58}$$

where $\Gamma^\alpha_{\lambda\mu}$, $\Delta^\alpha_{\lambda\mu}$ are arbitrary functions of $x_1, \cdots, x_4$. We regard (57) and (58) as defining an *affine connection* of the vectors $A^\alpha$ and $A_\alpha$ at $x$ with those at $x+dx$, or as defining *geodetic differentiation* of the $A^\alpha$ and $A_\alpha$. When the $A^\alpha$ and the $A_\alpha$ are transported along a curve such that $\delta A = 0$ we call the displacement a *parallel* or *geodetic displacement*. The geodetic differentials of tensors of higher order are formed in a manner analogous to their covariant differentials. Thus if $a^{\lambda\mu}$ is a tensor of order 2

$$\delta a^{\lambda\mu} = \sum_\alpha Q^{\lambda\mu}_\alpha dx_\alpha ,$$

where

$$Q^{\lambda\mu}_\alpha = \frac{\partial a^{\lambda\mu}}{\partial x_\alpha} + \sum_\beta a^{\lambda\mu} \Gamma^\mu_{\beta\alpha} + \sum_\beta a^{\beta\alpha} \Gamma^\lambda_{\beta\alpha}.$$

In Riemann's geometry $Q^\alpha_{\lambda\mu} = 0$; while in Weyl's,

$$Q^{\lambda\mu}_\alpha = a^{\lambda\mu} Q_\alpha .$$

The $\Gamma^{\alpha}_{\lambda\mu}$ and $\Delta^{\alpha}_{\lambda\mu}$ are not tensors but

$$C^{\alpha}_{\lambda\mu}=\Gamma^{\alpha}_{\lambda\mu}+\Delta^{\alpha}_{\lambda\mu}=\epsilon^{\alpha}_{\lambda}C_{\mu},\ \epsilon^{\alpha}_{\lambda}=0 \text{ or } 1, \text{ as } \alpha\neq\lambda \text{ or } =\lambda,$$

$$S^{\alpha}_{\lambda\mu}=\tfrac{1}{2}(\Gamma^{\alpha}_{\lambda\mu}-\Gamma^{\alpha}_{\mu\lambda})$$

are, as their labels indicate, tensors of order 3. In the geometry of Riemann, Weyl, and Eddington the $C^{\alpha}_{\lambda\mu}$ and $S^{\alpha}_{\lambda\mu}$ are 0. By means of $a^{\lambda\mu}$, $Q^{\alpha}_{\lambda\mu}$, $C_{\mu}$ and $S^{\alpha}_{\lambda\mu}$ the parameters $\Gamma$ and $\Delta$ can be simply expressed, and by making various assumptions different geometries arise.

If we displace geodetically the elementary vector $A^{\alpha}$ around a small circuit, we find that the $A^{\alpha}$ have changed by

$$\Delta A_{\alpha}=\frac{1}{2}\sum_{\lambda\mu\nu} B^{\alpha}_{\lambda\mu\nu}A^{\lambda}dS^{\mu\nu},$$

where $dS^{\mu\nu}$ is an element of area and $B^{\alpha}_{\lambda\mu\nu}$ is what $R^{\alpha}_{\lambda\mu\nu}$ in (23) becomes when the Christoffel symbols are replaced by the $\Gamma$'s which enter (57). Contraction of the $B^{\alpha}_{\lambda\mu\nu}$ gives us $B_{\lambda\mu}$ analogous to (24).

If we define the length $l$ of an elementary vector $A^{i}$ by

$$l^2=\sum_{ij}a_{ij}A^{i}A^{j},$$

we find on displacing geodetically $A^{i}$ from $x$ to $x+dx$, that $l^2$ has changed by the amount given in (36), when we replace the Christoffel symbols which figure there by the $\Gamma$'s of (57). In passing we may remark that in Weyl's theory

$$K_{\lambda_{\mu},k}=a_{\lambda_{\mu}}\varphi_{k}.$$

Let us see how Einstein uses this theory in his last paper (1925). He takes $C_{\mu}=0$; there are thus 64 $\Gamma$'s and 16 $a$'s to determine. To effect this he sets

$$g_{\lambda\mu}=\frac{a_{\lambda\mu}}{\sqrt{-a}},$$

and introduces the scalar density

$$H=\sum_{\lambda\mu}g^{\lambda\mu}B_{\lambda\mu}.$$

He then assumes that all the variations of the integral

$$\int H dx_1 dx_2 dx_3 dx_4$$

relative to the Γ's and $g$'s, regarded as independent variables, vanish. This gives $64+16=80$ equations to determine the 80 unknown $g$'s and Γ's.

11. *The Geometry of Paths.* In the foregoing, we have generalized the geometry of Riemann by generalizing the notion of parallel displacement; we may also proceed along another route as Eisenhart and Veblen (1922 et seq.) have shown. We are led to similar results but from another point of view, which, as these authors have shown, has important advantages.

As above we take the $n^3$ functions $\Gamma^i_{\lambda\mu}$ of $x_1, \cdots, x_n$ and write down the system of $n$ differential equations

$$\frac{d^2x_i}{ds^2} + \sum_{\lambda\mu} \Gamma^i_{\lambda\mu} \frac{dx_\lambda}{ds} \frac{dx_\mu}{ds} = 0 \ , \quad (i=1,2, \cdots ,n) \ , \qquad \Gamma^i_{\lambda\mu} = \Gamma^i_{\mu\lambda} \ .$$

These define a family of "curves called paths which like straight lines in euclidean space serve as a means of finding one's way about." These paths are geodesics in a Riemann space when there exists a metric $ds^2$ whose coefficients satisfy (21). In the general case the $\Gamma^i_{\lambda\mu}$ may be used to define an affine connection as in (57) while (58) is determined by taking $\Delta^i_{\lambda\mu} = -\Gamma^i_{\lambda\mu}$.

12. *Conclusions.* Let us bring this paper to a close by making a few remarks of a general nature. We began by describing what may be called the naive view of space. What space is *per se* is not discussed; its properties are codified in the geometry of Euclid. This geometry was taken over *in toto* by the physicist, astronomer, and engineer as the foundation of their science. Now Euclid's geometry had one weak spot, the notorious "fifth axiom." D'Alembert (1767) called it "*le scandale des éléments de la géométrie.*" All attempts to prove this axiom, which seemed more like

a proposition than a true axiom, were in vain. Finally Lobachevsky and Bolyai (1823–1833) showed how a consistent geometry might be constructed in which the parallel axiom did not hold. With this discovery a two thousand year period in geometry is closed and a new epoch begins. At first the significance of the Lobachevsky and Bolyai work was two fold.

1) It showed that the parallel axiom could not be proved and further attempts to do so were doomed to failure.

2) It bore heavily against some aspects of Kant's philosophy, particularly as to the a priori character of euclidean geometry.

Very gradually another view of space took shape. Gauss in a letter to Bessel (1830) writes "In all humility we must admit ⋯ that space has a reality independent of our mind and that we cannot lay down all its axioms".

This view of space we have seen was further developed by Riemann, Helmholtz, and Clifford (1854–1870).

We may characterize it briefly thus: Our sense perceptions, our daily experiences furnish the mind crude material which is worked up by it. One of its products is our notion of space. In the older view it was naively believed that our knowledge of this space, as far as its essential properties were concerned, was complete, and was embodied in the geometry of Euclid. Not so in the present view. According to this, the data of experience must be refined or idealized and rendered definite before a science of space, i.e., geometry, is possible. This idealization is effected by laying down certain definitions and axioms. A model of space is constructed in the same sense as Newton's mechanics is a mechanical model of the real world of mass and force, or Huygens undulatory theory of light is a model of the real phenomena of light. These models we call *abstract geometries*. Euclid's geometry is one, the geometry of Lobachevsky and Bolyai is another so are the sphero-elliptic geometry of Riemann, the elliptic geometry of Klein and Newcomb. There are many others. Which of these is most truly in accord with physical space must be found out by experience. But as soon as we begin to measure, we fall into a bottomless morass of physical theories; does our measuring rod change as we move it about, is the path of light a euclidean straight, are our mirrors euclidean planes; in short, geometry and physics seem indissolubly bound together. It must be remarked, however, that these views were not held by the rank and file of the scientific world. Indeed the attitude of scientists was more like the attitude of the scholastic world as whether the earth was flat or a sphere.

This state of mind was rudely shaken by the appearance of the theory of relativity (1914 et seq.). This theory not only requires a space whose structure is in constant flux, changing with every displacement of gravitational or electric masses, but it also gives the means of experimentally determining this structure.

A most significant fact in the latest development of the theory of relativity is the preponderant role played by geometry. There is no one theory of relativity and to-day geometers are discovering, as we have seen, new geometries, one of which may prove the best adapted to give an intelligible picture of the physical world from the standpoint of relativity.

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