THE HEAVISIDE OPERATIONAL CALCULUS*

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The Heaviside operational calculus is a systematic method, originated and developed by Oliver Heaviside, for the solution of systems of linear differential equations with constant coefficients, and linear partial differential equations of the type of the *wave equation.* Its important applications in mathematical physics are to the dynamics of small oscillations, the Fourier theory of heat flow and to electrical transmission theory. It was, in fact, in connection with problems of the latter class that the operational calculus was developed, and in the solution of such problems it is an instrument of remarkable directness, simplicity and power.†

The operational calculus may be described with sufficient generality for our present purposes, in connection with the solution of the system of linear differential equations

$$(1)\qquad \begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots+a_{1n}x_n=f_1(t)\,, \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots+a_{2n}x_n=f_2(t)\,, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\cdots+a_{nn}x_n=f_n(t)\,, \end{cases}$$

where the coefficient a_{jk} has the form

$$(2)\qquad a_{jk}=\alpha_{jk}+\beta_{jk}\frac{d}{dt}+\gamma_{jk}\frac{d^2}{dt^2}+\cdots.$$

$\alpha, \beta, \gamma, \cdots$ being constants. We require the solution for $x_1, x_2, \cdots, x_n$ for the following boundary conditions: The known functions $f_1, \cdots, f_n$ and the dependent variables $x_1, \cdots, x_n$ are identically zero for $t<0$. In the language of dynamics, the "forces" $f_1, \cdots, f_n$ are applied to a system

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† For a brief discussion of the Heaviside operational calculus from the point of view adopted in this paper, see Carson, *The Heaviside Operational Calculus*, BELL SYSTEM TECHNICAL JOURNAL, Nov., 1922.

characterized by the coordinates $x_1, \cdots, x_n$, when in a state of equilibrium. These boundary conditions are extremely important to the physicist, and it is to the solution of this boundary value problem that the operational calculus applies.

In view of the linear character of the equations, we may, without loss of essential generality, set all the f functions equal to zero except one, say $f_1(t)$, and write the equations as

$$(3)\qquad \begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=f_1(t)\ , \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=0\ , \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=0\ . \end{cases}$$

Now consider the auxiliary system of equations

$$(4)\qquad \begin{cases} a_{11}h_1+a_{12}h_2+\cdots+a_{1n}h_n=1\ , \quad t\geqq 0\ , \\ a_{21}h_1+a_{22}h_2+\cdots+a_{2n}h_n=0\ , \\ \cdots\cdots\cdots\cdots\cdots \\ a_{n1}h_1+a_{n2}h_2+\cdots+a_{nn}h_n=0\ , \end{cases}$$

in the auxiliary variables $h_1, \cdots, h_n$. As in (1) and (2), both sides of the equations are supposed to be identically zero for $t<0$.

For the equilibrium boundary conditions the solution of (4) supplies the solution of (3) by virtue of the formula

$$(5)\qquad x_i=\frac{d}{dt}\int_0^t f(t-\tau)h_i(\tau)d\tau\ , \qquad (i=1,2,\cdots,n)\ .$$

This formula is easily established in a number of ways; perhaps the simplest, and at the same time the most general, method of proof is to regard $f(t)$ as the limit of a summation of the form

$$\sum\phi(t-n\delta t)\delta f_n\ ,$$

where $\phi(t)$ is defined as a function which is zero for $t<0$ and unity for $t\geqq 0$. The solution of (4) applies to this case and passing to the limit ($\delta t\rightarrow 0$) formula (5) results by usual processes.

It is to equations (4) that the operational calculus applies; Heaviside did not explicitly consider equations (3) in their completely general form.

Heaviside's treatment of equations (4) proceeded as follows: Replacing the differential operator d^n/dt^n by p^n, the differential equations become formally algebraic and yield the *operational equation*

$$h_i=\frac{1}{H_i(p)}, \qquad (i=1, 2, 3, \cdots, n). \tag{6}$$

I have called (6) an equation; as a matter of fact it is not an equation except in a purely symbolic sense, since the left hand side is a function of t and the right hand side a function of p. Heaviside's point of view, however, was that (6) is the full equivalent of (4), and that the functional form of $H(p)$ contains all the information necessary for the explicit solution.

From this point on, however, his method was purely inductive; from the known solution of specific problems, he inferred rules for expanding and interpreting the operational equation (6), and for converting it into the explicit solution. The body of rules and formulas so arrived at is termed the operational calculus.

Instead of following Heaviside's inductive methods, we shall now take advantage of the following theorem.

The operational equation

$$h_i=1/H_i(p)$$

is simply the symbolic or short hand equivalent of the integral equation

$$\frac{1}{pH_i(p)}=\int_0^\infty h_i(t)e^{-pt}dt\ ,$$

which is valid for all values of p in the right hand half of the complex plane. This uniquely determines a continuous function $h_i(t)$ in terms of $H_i(p)$. From this integral equation the rules and formulas of the operational calculus are directly and immediately deducible.

The equation (dropping subscripts)

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt \tag{7}$$

is an integral equation of the Laplace type. Beyond the proof that it determines a unique continuous function $h(t)$, curiously little work appears to have been done on it from the standpoint of analysis. At present, however, we are concerned with it only as an instrument for the solution of the differential equations (4).

The proof of the preceding theorem is easily arrived at as follows: Referring to equation (3) suppose that $f(t)=e^{pt}(t\geqq 0)$ where *p is a positive real constant* or complex with real part positive. The solution of equations (3) with the subscripts dropped) is then of the form

$$x=\frac{e^{pt}}{H(p)}+y(t) \ , \tag{8}$$

where $y(t)$ is the complementary solution, so constructed as to satisfy the equilibrium boundary condition. But by (5), we have also

$$x=\frac{d}{dt}\int_0^t e^{p(t-\tau)}h(\tau)d\tau \ , \tag{9}$$

whence, by direct equation,

$$\frac{e^{pt}}{H(p)}+y(t)=\frac{d}{dt}\int_0^t e^{p(t-\tau)}h(\tau)d\tau \ .$$

Carrying out the indicated differentiation and simplifying, we get

$$\frac{1}{H(p)}+y(t)e^{-pt} = h(t)e^{-pt}+p\int_0^t h(\tau)e^{-\tau p}d\tau \ .$$

Finally, setting $t=\infty$,

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}dt \ ,$$

provided the real part of p is positive. This establishes formula (7) as an identity for all values of p in the right hand half of the complex plane.

Before proceeding with an application of the foregoing integral equation, it should be remarked that while the operational calculus has been elucidated in connection with a set of differential equations involving a finite number of variables, it is not so limited in its applications. The derivations of formula (5) and the integral equation (7) are quite general and depend only on the linear character of the differential equations and the linear invariable character of the dynamic system which they describe. Consequently the resulting formulas are applicable also when the number of variables is infinite (cases of which occur in important electrotechnical problems) and to linear partial differential equations of the type of the *wave equation*, the *equation of the conduction of heat*, and the *telegraph equation*.*

We now turn to a discussion of the integral equation

$$\frac{1}{pH(p)}=\int_0^\infty h(t)e^{-pt}dt$$

as a means of deriving the rules of the operational calculus, and as an instrument for the direct solution of problems. My own work on this problem has been directed as follows: (1) guided by the inductive work of Heaviside on the operational equation, to develop general rules for the expansion and transformation of the integral equation, leading to general types of expansion solutions. (2) The accumulation of a table of definite integrals of the type

$$\int_0^\infty \phi(t)e^{-pt}dt\,.$$

It hardly needs to be pointed out, in view of the symbolic equivalence of the operational and the integral equations, that

* See Whittaker and Watson, *Modern Analysis*, p. 386, for a discussion of these equations.

the evaluation of an infinite integral of this type immediately furnishes the solution of the corresponding operational equation. The table of integrals which has been collected has proved quite valuable in technical problems.

We shall now deduce two expansion solutions of the integral equation, corresponding to two important forms of solution of the operational equation first given by Heaviside.

The Power Series Solution. Let us suppose, as is the case in a very large class of physical problems, that the function $1/H(p)$ admits of formal asymptotic expansion; thus

$$1/H(p) \sim \sum_{0}^{\infty} a_n/p^n \,. \tag{10}$$

With the convergence or divergence of such an expansion we have no concern. It must, however, satisfy Poincaré's definition of an asymptotic expansion.*

Now starting with the integral equation (7) and integrating by parts, we get

$$\frac{1}{H(p)} = h(0) + \int_0^\infty e^{-pt} h^{(1)}(t)dt \,,$$

where $h^{(n)}(t) = (d^n/dt^n)h(t)$. Now let p approach infinity; in the limit the integral vanishes and, by virtue of the asymptotic expansion (10), the left hand side becomes a_0. Consequently

$$h(0) = 1/H(\infty) = a_0.$$

Integrating again by parts, we get

$$p\left(\frac{1}{H(p)} - a_0\right) = h^{(1)}(0) + \int_0^\infty e^{-pt} h^{(2)}(t)dt \,.$$

Now let p again approach infinity; in the limit the integral vanishes and the left hand side by virtue of (10) becomes a_1. Consequently

$$h^{(1)}(0) = a_1.$$

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* See Whittaker and Watson, *Modern Analysis*, p. 151.

Proceeding in this way we establish the relation

$$h^{(n)}(0) = a_n.$$

It follows at once that, if we *assume* that a power series expansion of $h(t)$ exists, it is given by

$$h(t) = \sum_0^\infty a_n t^n / n! \tag{11}$$

We thus arrive at the Heaviside rule:

In the operational equation,

$$h = 1/H(p),$$

expand $1/H(p)$ *in inverse powers of* p:

$$1/H(p) \sim a_0 + a_1/p + a_2/p^2 + \cdots. \tag{10}$$

The explicit solution for $h(t)$ *results by replacing* $1/p^n$ *by* $t^n/n!$ *in the asymptotic expansion, so that*

$$h(t) = a_0 + a_1 t/1! + a_2 t^2/2! + \cdots \tag{11}$$

*is the required power series solution.**

It may be remarked in passing that this type of solution, while extremely direct and always possible in the case of systems having a finite number of degrees of freedom, is of minor practical importance unless the power series can be recognized and summed. Furthermore this form of solution does not exist in many important technical problems.

The Solution in Terms of Normal Vibrations. It is known from the usual elementary theory of linear differential equations that the solution of equations (4) may be written in the form

$$h_i = C_{i0} + \sum_0^m C_{ij} e^{p_j t}, \qquad (i = 1, 2, \cdots, n) \tag{12}$$

where $p_1, p_2, \cdots, p_m$ are the roots of the equation

$$H_i(p) = \frac{D(p)}{M_i(p)} = 0, \tag{13}$$

* It is interesting to note that the series (11) is Borel's associated function of the series (10), and that the infinite integral is the Borel sum of the series (10).

and the C's are constants of integration which must be so chosen as to satisfy the system of differential equations and the imposed boundary conditions. Assuming that these roots can be located, the problem of determining the constants of integration, while formally straightforward, is extremely troublesome in practice when the number of degrees of freedom is large. It will now be shown that, assuming the roots $p_1, p_2, \cdots$ to have been evaluated, the constants of integration can be evaluated from the integral equation with extreme simplicity. Dropping the double subscript, we write the known form of solution as

$$h = C_0 + \sum C_j e^{p_j t}, \tag{14}$$

and substitute in the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty h(t) e^{-pt} dt. \tag{15}$$

Integrating term by term we get

$$\frac{1}{pH(p)} = \frac{1}{p} C_0 + \sum \frac{C_j}{p - p_j}, \tag{16}$$

where, it will be recalled, p_j is a root* of $H(p)$.

Multiplying through by $pH(p)$ we get

$$C_0 H(p) + \sum \frac{pH(p)}{p - p_j} C_j = 1. \tag{17}$$

We now introduce restrictions which obtain in physical problems: $H(p)$ has no zero root, no repeated roots, and $1/H(p)$ is a proper fraction of the form $M(p)/D(p)$ where the numerator and denominator are prime to each other, and the numerator of lower order in p than the denominator.

Now if we set $p = 0$, the summation vanishes, and we get

$$C_0 = 1/H(0).$$

* In all physical problems, the real part of p_j is negative, so that the integration of (15) is valid.

Next set p equal to p_j+q, and then let q approach zero. Since $H(p_j)=0$, it follows at once that (17) reduces to

$$p_jH'(p_j)C_j=1\ ,$$

whence

$$C_j = \frac{1}{p_jH'(p_j)}\ , \tag{18}$$

where

$$H'(p_j) = \left[\frac{d}{dp}H(p)\right]_{p=p_j} .$$

We thus arrive at the solution

$$h(t) = \frac{1}{H(0)}+\sum\frac{e^{p_jt}}{p_jH'(p_j)}\ , \tag{19}$$

first stated by Heaviside without proof. It was apparently arrived at inductively from the operational equation

$$h=1/H(p)\ .$$

This is a beautifully compact solution, and of first-rate importance in many technical problems.

It is quite beyond the scope of this paper to discuss in detail the appropriate expansion and transformations of the integral equations, each having its counterpart in the corresponding operational equation, which are useful when the operational equation does not admit of direct solution as it stands. A few theorems, immediately derivable from the integral equation, may be of interest.

1. *If two functions h and g are defined by the operational equations*

$$h=1/H(p)\ ,\quad \textit{and}\quad g=1/pH(p)\ ,$$

then

$$g(t)=\int_0^t h(t)dt\ .$$

2. *If h and g are defined by the operational equations*

$$h=1/H(p)\ ,\qquad g=p/H(p)\ ,$$

then

$$g(t)=\frac{d}{dt}h(t)\ ,$$

provided $h(0)=0$.

3. *If h and g are defined by the operational equations*

$$h=1/H(p)\ ,\qquad g=1/H(p+\lambda)\ ,$$

then

$$g(t)=\left(1+\lambda\int_0^t dt\right)e^{-\lambda t}h(t)\ .$$

4. *If h and g are defined by the operational equations*

$$h=1/H(p)\ ,\qquad g=e^{-\lambda p}/H(p)\ ,$$

then

$$\begin{aligned} g&=0 \quad \textit{for} \quad t<\lambda\ ,\\ &=h(t-\lambda) \quad \textit{for} \quad t\geqq\lambda\ \cdot \end{aligned}$$

5. *Let h be defined by the operational equation*

$$h=1/H(p)\ ,$$

and suppose that $H(p)$ *can be factored in the form*

$$H(p)=H_1(p)\cdot H_2(p)\ .$$

Let the auxiliary functions h_1 *and* h_2 *be defined and determined by the auxiliary operational equations*

$$h_1=1/H_1(p)\ ,\qquad h_2=1/H_2(p)\ ;$$

then

$$\begin{aligned} h(t)&=\frac{d}{dt}\int_0^t h_1(t-\tau)h_2(\tau)d\tau\\ &=\frac{d}{dt}\int_0^t h_2(t-\tau)h_1(\tau)d\tau\ . \end{aligned}$$

The following theorem is of considerable theoretical interest, as it formally extends the methods of the operational calculus to the case of arbitrary functions $f(t)$ as compared with the "unit" function of equations (4).

6. *If the function* $f(t)$ *of equations* (3) *is such that the infinite integral*

$$\int_0^\infty f(t)e^{-pt}dt$$

can be evaluated and is equal to $1/pF(p)$, *then* $x(t)$ *of equations* (3) *is given symbolically by the operational equation*

$$x = \frac{1}{F(p) \cdot H(p)} = \frac{1}{X(p)} ,$$

and by the corresponding integral equation

$$\frac{1}{X(p)} = \int_0^\infty x(t)e^{-pt}dt .$$

This integral equation corresponds precisely with (7), *and is solvable by the same general procedure.*

The Telegraph Equation. The *telegraph equation* formulates the propagation of current and voltage along the transmission line. It will serve as an excellent example of the utility of the operational calculus and the integral equation.

We suppose that the transmission line extends along the positive axis and is energized by an electromotive force applied at $x=0$. The differential equations of current I and voltage V are

$$\left(L\frac{d}{dt} + R\right) I = -\frac{\partial}{\partial x} V ,$$

$$\left(C\frac{d}{dt} + G\right) V = -\frac{\partial}{\partial x} I ,$$

where L, R, C and G are the distributed constants of the line, the inductance, resistance, capacity and leakage per unit length.

Replacing d/dt by the symbol p, we get

$$(Lp+R)I=-\frac{\partial}{\partial x}V\ ,$$

$$(Cp+G)V=-\frac{\partial}{\partial x}I\ ,$$

thus formally eliminating the variable time t. Successive elimination of V and of I give the equations

$$\gamma^2 I=\frac{\partial^2}{\partial x^2}I\ ,$$

$$\gamma^2 V=\frac{\partial^2}{\partial x^2}V\ ,$$

where $\gamma^2=(Lp+R)(Cp+G)$. These equations are satisfied by solutions of the form

$$V=Ae^{-\gamma x}+Be^{\gamma x}\ ,$$

$$I=\frac{Cp+G}{\gamma}(Ae^{-\gamma x}-Be^{\gamma x})\ ,$$

where A and B are constants of integration.

We now assume that the line is infinitely long so that the reflected wave vanishes. We also assume a voltage $V_0(t)=V_0$ applied at $x=0$. The equations then become

$$V=V_0e^{-\gamma x}\ ,$$

$$I=\frac{Cp+G}{\gamma}V_0e^{-\gamma x}\ .$$

Finally, if we assume that V_0 is zero for $t<0$ and unity for $t\geqq 0$, we have the *operational equations*

$$V=e^{-\gamma x}\ ,$$

$$I=\frac{Cp+G}{\gamma}e^{-\gamma x}\ .$$

Let us write

$$\gamma^2 = \frac{1}{v^2}[(p+\rho^2) - \sigma^2] ,$$

where

$$v^2 = 1/LC , \quad \rho = \frac{R}{2L} + \frac{G}{2C} , \quad \sigma = \frac{R}{2L} - \frac{G}{2C} ,$$

and let us consider the operational equation defining a new variable F:

$$F = \frac{p}{v\gamma} e^{-\gamma x} = \frac{p}{\sqrt{(p+\rho)^2 - \sigma^2}} e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}} . \tag{20}$$

It follows at once from the operational rules discussed above (Theorems 1 and 2) that

$$I = v\left(C + G\int_0^t dt\right)F , \tag{21}$$

$$V = -v\int_0^t \frac{\partial F}{\partial x} dt . \tag{22}$$

Our problem is thus reduced to evaluating the function F, as defined by the operational equation (20) and the corresponding integral equation

$$\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2 - \sigma^2}} = \int_0^\infty F(t) e^{-pt} dt . \tag{23}$$

The solution of this integral equation is deducible from the known definite integral

$$\int_\lambda^\infty e^{-pt} J_0(\sqrt{t^2 - \lambda^2}) dt = \frac{e^{-\lambda\sqrt{p^2+1}}}{\sqrt{p^2+1}} . \tag{24}$$

This, regarded as an integral equation, determines a function which is zero for $t<\lambda$ and has the value $J_0(\sqrt{t^2-\lambda^2})$ for $t\geqq\lambda$, J_0 being the Bessel function of the first kind of order zero. In order to solve (23) by aid of (24) we proceed to transform the latter as follows:

(1) Let $\lambda p=q$ and $t/\lambda=t_1$. Substitute in (24) and then replace q by p and t_1 by t, in order to retain our original symbolism. We get

$$\frac{e^{-\sqrt{p^2+\lambda^2}}}{\sqrt{p^2+\lambda^2}}=\int_1^\infty e^{-pt}J_0(\lambda\sqrt{t^2-1})dt\ . \tag{25}$$

(2) Making the substitution $p=q+\mu$ in (25) and then replacing q by p, we get

$$\frac{e^{-\sqrt{(p+\mu)^2+\lambda^2}}}{\sqrt{(p+\mu)^2+\lambda^2}}=\int_1^\infty e^{-pt}e^{-\mu t}J_0(\lambda\sqrt{t^2-1})dt\ . \tag{26}$$

(3) Making the substitution

$$p=\frac{x}{v}q\ ,\ \text{and}\ t=\frac{v}{x}t_1\ ,$$

in (26), and ultimately replacing q by p and t_1 by t, we get

$$\frac{e^{-\frac{x}{v}\sqrt{(p+\mu)^2+\lambda^2}}}{\sqrt{(p+\mu)^2+\lambda^2}}=\int_{x/v}^\infty e^{-pt}e^{-\mu t}J_0(\lambda\sqrt{t^2-x^2/v^2})dt \tag{27}$$

where λ and μ are parameters at our disposal, except that, as yet, they are restricted to positive real values.

(4) Now if we compare (27) with the integral equation (23) it will be observed that the left sides become identical if we set $\mu=\rho$ and $\lambda=i\sigma=\sigma\sqrt{-1}$. (This last operation is justified because $\sigma\leqq\rho$.) Introducing these relations we get finally

$$\frac{e^{-\frac{v}{x}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}=\int_{x/v}^\infty e^{-pt}e^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2})dt\ , \tag{28}$$

where $I_0(x)=J_0(ix)$. Consequently it follows that

$$(29)\quad \begin{cases} F=0 \quad \text{for} \quad t<x/v \\ \quad =e^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2}) \quad \text{for} \quad t\geqq x/v\,. \end{cases}$$

It now follows from (21), (22) and (29) by straightforward procedure, that

$$(30)\quad \begin{cases} I=0 \quad \text{for} \quad t<x/v \\ \quad =\sqrt{\dfrac{C}{L}}e^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2}) \\ \qquad +vG\displaystyle\int_{x/v}^{t} e^{-\rho\tau}I_0(\sigma\sqrt{\tau^2-x^2/v^2})\,d\tau\,, \qquad t\geqq x/v; \end{cases}$$

$$(31)\quad \begin{cases} V=0 \quad \text{for} \quad t<x/v \\ \quad =e^{-\rho x/v}+\dfrac{\sigma x}{v}\displaystyle\int_{x/v}^{t}\frac{e^{-\rho t}I_1(\sigma\sqrt{\tau^2-x^2/v^2})}{\sqrt{\tau^2-x^2/v^2}}\,d\tau, \qquad t\geqq x/v. \end{cases}$$

The foregoing is believed to be an excellent example of the value of the operational calculus and in particular of the advantage attaching to the recognition of the integral equation identity. The directness and simplicity of the solution, as derived above, as compared with its derivation by classical methods, is noteworthy. At the same time it must be admitted that a direct solution from the operational equation, without recognizing the integral equation identity, presents formidable difficulties. Heaviside's own attack on this problem from the operational equation, while distinguished by extraordinary ingenuity and almost uncanny intuition, can hardly be regarded as entirely satisfactory.*

However, without a knowledge of the integral identity (24) on which the preceding solution is based, it is possible to derive a series solution as follows. The method will be sketched for the operational equation

$$V=e^{-\gamma x}$$

* See Heaviside's *Electromagnetic Theory*, vol. II, p. 290 et seq.

for the voltage. The procedure for the current is the same and will require no explanation. The method of solution depends on Theorem 4, given above. Since

$$\gamma x = \frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}$$
$$= \frac{x}{v} p \sqrt{\left(1+\frac{\rho}{p}\right)^2 - \frac{\sigma^2}{p^2}},$$

we can write the operational equation as

$$V = e^{-\frac{x}{v}p\sqrt{\left(1+\frac{\rho}{p}\right)^2-\frac{\sigma^2}{p^2}}}$$
$$= e^{-\frac{x}{v}p\left(1+\frac{\rho}{p}+\frac{a_1}{p^2}+\frac{a_2}{p^3}+\cdots\right)},$$

by the binomial expansion of the exponent. This is formally reducible to the form

$$V = e^{-\frac{x}{v}p} \cdot e^{-\frac{\rho v}{\rho}}\left(1+\frac{A_1}{p}+\frac{A_2}{p^2}+\frac{A_3}{p^3}+\cdots\right).$$

A direct application of Theorem 4 now gives

$$V = 0 \text{ for } t < x/v$$
$$= e^{-\frac{\rho}{v}x}\left\{1 + A_1\frac{t-x/v}{1!} + A_2\frac{(t-x/v)^2}{2!} + A_3\frac{(t-x/v)^3}{3!} + \cdots\right\}.$$

It is easily verified that this series solution in the "retarded" time $t-x/v$ is absolutely convergent and identical with the expansion of the solution (21).

The Asymptotic Solution of Operational Equations. An extremely interesting and important part of the operational calculus relates to the derivation of asymptotic expansions directly from the operational equation. A study of the many problems for which Heaviside obtains asymptotic expansions shows that they may be divided into two classes: (1) those of which the operational equation is of the form

$$\text{(I)} \qquad I=\sqrt{p}F(p) \ ,$$

and (2) those of which the operational equation is of the form*

$$\text{(II)} \qquad h=\phi(\sqrt{p}) \ .$$

Heaviside himself gives no justification or proof of his expansion solutions. He does not formally distinguish between the two classes, and he gives no information regarding the asymptotic character of the series. While a completely satisfactory theory of these expansions has not as yet been worked out, the application of the Laplace integral equation to their investigation throws a great deal of light on the problem, and at least reduces it to a form to which the orthodox processes of analysis are applicable.

We start with the type of problem (class 1), which is symbolically formulated by operational equations of the type

$$(32) \qquad h=\sqrt{p}\, F(p) \ ,$$

of which the corresponding integral equation is

$$(33) \qquad \frac{F(p)}{\sqrt{p}}=\int_0^\infty h(t)e^{-pt}dt \ ,$$

and we suppose that $F(p)$ admits of the power series expansion

$$(34) \qquad a_0+a_1p+a_2p^2+ \cdots \ .$$

For this problem the Heaviside rule is as follows.

If the operational equation

$$h=1/H(p)$$

* More generally $h=\phi(p^k\sqrt{p})$ where k is a positive integer. The case where $k=0$, however, illustrates Heaviside's procedure with sufficient generality.

can be expanded in the form

$$h=(a_0+a_1p+a_2p^2+a_3p^3+\cdots)\sqrt{p}\ ,$$

the explicit solution, in the form of an asymptotic series, is obtained by replacing the symbol $\sqrt{p}$ *by* $1\sqrt{\pi t}$, *and* p^n *by* d^n/dt^n, *whence*

$$h(t)\sim\left(a_0+a_1\frac{d}{dt}+a_2\frac{d^2}{dt^2}+\cdots\right)\frac{1}{\sqrt{\pi t}} \tag{35}$$

$$\sim\frac{1}{\sqrt{\pi t}}\Big(a_0-a_1/2t+1\cdot 3a_2/(2t)^2 - 1\cdot 3\cdot 5a_3/(2t)^3+\cdots\Big)\ . \tag{35a}$$

We now proceed to derive this rule from the integral equation (33), and to discuss its limitations.

We assume the existence of an auxiliary function $f(t)$, defined and determined by the auxiliary integral equation

$$F(p)=\int_0^\infty f(t)e^{-pt}dt\ . \tag{36}$$

Now since

$$\frac{1}{\sqrt{p}}=\int_0^\infty e^{-pt}\frac{dt}{\sqrt{\pi t}}\ ,$$

it follows from Theorem 5 above that $h(t)$, as defined by (33), is given by

$$h(t)=\frac{1}{\sqrt{\pi}}\int_0^t\frac{f(\tau)}{\sqrt{t-\tau}}\,d\tau\ . \tag{37}$$

We observe that if (36) is differentiated repeatedly with respect to p and p set equal to zero, then, by virtue of the expansion (34),

$$a_n=(-1)^n\int_0^\infty\frac{t^n}{n!}f(t)dt\ ,\qquad (n=1,\,2,\,\cdots)\ . \tag{38}$$

It will be noted, in this connection, that this expansion presupposes the convergence of the infinite integrals (38) for all values of n and therefore imposes severe restrictions on $f(t)$ and hence on $F(p)$. We shall suppose that these restrictions are satisfied.

We are now prepared to prove the following theorem.

A necessary and sufficient condition for the validity of the Heaviside asymptotic expansion (35) *of the operational equation*

$$h=F(p)\sqrt{p}\ ,$$

is that the definite integral (37) *shall be asymptotically representable by the series*

$$(39)\quad \left\{ \begin{aligned} \frac{1}{\sqrt{\pi t}}\bigg\{ \int_0^\infty f(t)dt &+ \frac{1}{2t}\int_0^\infty \frac{t}{1!}f(t)dt \\ &+ \frac{1\cdot 3}{(2t)^2}\int_0^\infty \frac{t^2}{2!}f(t)dt \\ &+ \frac{1\cdot 3\cdot 5}{(2t)^3}\int_0^\infty \frac{t^3}{3!}f(t)dt + \cdots \bigg\}. \end{aligned} \right.$$

This theorem is an immediate consequence of the preceding analysis, for by (38) the series (39) is simply

$$\frac{1}{\sqrt{\pi t}}\,(a_0-a_1/2t+1\cdot 3a_2/(2t)^2-\ \cdots)\ ,$$

whence, by (37),

$$h(t)\sim\frac{1}{\sqrt{\pi t}}\,(a_0-a_1/2t+1\cdot 3a_2/(2t)^2-\ \cdots)\ ,$$

which is simply the Heaviside expansion (35a). We have thus succeeded in reducing the problem of the asymptotic expansion from the operational equation to that of the asymptotic expansion of the definite integral

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\,d\tau\ ,$$

where $f(t)$ is defined and determined by the integral equation

$$F(p)=\int_0^\infty f(t)e^{-pt}dt \ ,$$

If we write the definite integral in the form

$$\frac{1}{\sqrt{\pi t}}\int_0^t d\tau \cdot f(\tau) \cdot (1-\tau/t)^{-1/2} \ , \tag{40}$$

we see at once that the series (39) is obtained by expanding $(1-\tau/t)^{-1/2}$ by the binomial theorem, replacing the upper limit in integration by infinity, and integrating term-by-term.

That the series (39) does represent the behavior of the function (37) for sufficiently large values of t, provided $f(t)$ converges with sufficient rapidity, is evident qualitatively from the form of the definite integral. It can also be shown rigorously that the asymptotic representation of (37) by the series (39) is valid provided the integrals (38) exist. We thus arrive at the conclusions, deducible by rigorous processes, that the validity of the Heaviside rule stated above depends on the properties of the function $f(t)$, defined by the integral equation (36), and in particular on the existence of the infinite integrals (38). On the other hand, the precise sense in which the expansion asymptotically represents the solution will depend on the particular problem, and no general statement on this subject can be made.

We now take up the problem of the expansion solutions, usually divergent, of problems of class 2, formulated by operational equations of the form

$$h=\phi(\sqrt{p}) \ , \tag{41}$$

of which the corresponding integral equation is

$$\frac{1}{p}\phi(\sqrt{p}) =\int_0^\infty h(t)e^{-pt}dt \ . \tag{42}$$

If we suppose that $\phi(\sqrt{p})$ admits of power series expansion in the argument, we may write

$$(43) \qquad \phi(\sqrt{p}) = b_0 + b_1\sqrt{p} + b_2 p + b_3 p\sqrt{p} + b_4 p^2 + \cdots,$$

without, however, making any statement regarding the convergence of this expansion.

The Heaviside rule for this type of problem may be stated as follows.

If the operational equation $h = 1/H(p)$ *can be expanded in the form*

$$h = b_0 + b_1\sqrt{p} + b_2 p + b_3 p\sqrt{p} + \cdots,$$

discard all terms containing integral powers of p, *and write*

$$h = b_0 + (b_1 + b_3 p + b_5 p^2 + \cdots)\sqrt{p}\,.$$

The series solution is then obtained by replacing $\sqrt{p}$ *by* $1/\sqrt{\pi t}$ *and* p^n *by* d^n/dt^n, *whence,*

$$(44) \qquad h(t) = b_0 + \left(b_1 + b_3\frac{d}{dt} + b_5\frac{d^2}{dt^2} + \cdots\right)\frac{1}{\sqrt{\pi t}}$$

or

$$(45) \qquad h(t) = b_0 + \frac{1}{\sqrt{\pi t}}\left(b_1 - \frac{b_3}{2t} + \frac{1\cdot 3 b_5}{(2t)^2} - \frac{1\cdot 3\cdot 5 b_7}{(2t)^3} + \cdots\right).$$

In order to investigate this expansion solution, we construct the auxiliary functions $g, g_1, g_2, \cdots, g_n, \cdots$, defined by the following scheme:

$$g(t) = h(t) - b_0\,,$$

$$g_1(t) = g(t) - \frac{b_1}{\sqrt{\pi t}}\,,$$

$$g_2(t) = t\cdot g_1(t) + \frac{1}{2}\frac{b_3}{\sqrt{\pi t}}\,,$$

$$g_3(t) = t\cdot g_2(t) - \frac{1\cdot 3}{2^2}\frac{b_5}{\sqrt{\pi t}}\,,$$

$$g_4(t) = t\cdot g_3(t) + \frac{1\cdot 3\cdot 5}{2^3}\frac{b_7}{\sqrt{\pi t}}\,,$$

$$\cdots\cdots\cdots\cdots\cdots.$$

Successive substitutions in the integral equation (42) and repeated differentiation with respect to p, lead to the set of formulas

$$\int_0^\infty g(t)e^{-pt}dt \sim \frac{b_1}{\sqrt{p}} \quad \text{as} \quad p \to 0 \,,$$

$$\int_0^\infty t\cdot g_1(t)e^{-pt}dt \sim -\frac{b_3}{2\sqrt{p}} \quad \text{as} \quad p \to 0 \,,$$

$$\int_0^\infty t\cdot g_2(t)e^{-pt}dt \sim \frac{1\cdot 3}{2^2}\,\frac{b_5}{\sqrt{p}} \quad \text{as} \quad p \to 0 \,,$$

$$\int_0^\infty t\cdot g_3(t)e^{-pt}dt \sim -\frac{1\cdot 3\cdot 5}{2^3}\,\frac{b_7}{\sqrt{p}} \quad \text{as} \quad p \to 0 \,.$$

Now since

$$\int_0^\infty \frac{e^{-pt}}{\sqrt{\pi t}}dt = \frac{1}{\sqrt{p}} \,,$$

it follows from the preceding that the functions $g_1, g_2, g_3, \cdots$ all converge to zero as $1/t\sqrt{\pi t}$ as $t \to \infty$, provided $g(t)$ *contains no term which converges in an oscillatory manner.* It is this latter restriction, namely that $g(t)$ must contain no oscillatory term, say of the form

$$\frac{1}{\sqrt{\pi t}}\cos t \,,$$

which stands in the way of a satisfactory theory.

However, *assuming* that the auxiliary functions $g_1, g_2, g_3, \cdots$ converge to zero as $1/t\sqrt{\pi t}$ as $t \to \infty$, the foregoing leads at once to the formal asymptotic expansion (45). The same procedure, it can be remarked, is applicable to class I problems and gives formally the expansion (35a).

The foregoing cannot be regarded as a satisfactory discussion and is indeed almost as heuristic as Heaviside's own procedure. In addition to the theoretical defects it is unsatisfactory to the physicist who requires an automatic rule for deriving the asymptotic expansion from the operational equation by algebraic processes without investigation of auxiliary functions, or remainder terms, and in particular an estimate of the numerical error committed by stopping with any term of the series.* Some of these defects it is believed can be overcome; some are inherent, however, in the nature of the problem. For example, in some problems the series is asymptotic in the sense that the error is less than the last term retained; in others it is meaningless for a certain range of values of t, while in yet others, remarkably enough, the series is absolutely convergent for all finite values of t.

We shall now close this paper with a few examples of the series expansion just discussed. The first example will be the asymptotic solution of the operational equation

$$h=\frac{\sqrt{p}}{\sqrt{p+2\rho}}\,, \tag{46}$$

where ρ is a positive real parameter. The corresponding integral equation is

$$\frac{1}{\sqrt{p^2+2\rho p}}=\int_0^\infty h(t)e^{-pt}dt\,. \tag{47}$$

The physical problem which this represents is the current entering a transmission line of distributed constants L, R, C, in response to a unit e.m.f. applied at time $t=0$.

The solution of equation (47) is known. It is

$$h(t)=e^{-\rho t}I_0(\rho t)\,, \tag{48}$$

where I_0 is the modified Bessel function of the first kind and order zero.

Now return to the operational equation (46). It will be observed that it is of the form $h=F(p)\sqrt{p}$, and hence falls

* Heaviside appears to have regarded the series as asymptotic in the sense that the numerical error is less than the value of the last term retained. This, however, is certainly not true in general.

within class I. Expanding (46) in accordance with the Heaviside rule, we find

$$h=\left\{1-\frac{p}{4\rho}+\frac{1\cdot 3}{2!}\left(\frac{p}{4\rho}\right)^2-\frac{1.3.5}{3!}\left(\frac{p}{4\rho}\right)^3+\cdots\right\}\sqrt{p/2\rho}\ .$$

Now apply the Heaviside rule given above for converting this operational expansion into the explicit solution. We get

$$h(t)\sim\frac{1}{\sqrt{2\pi\rho t}}\left\{1+\frac{1}{8\rho t}+\frac{(1\cdot 3)^2}{2!(8\rho t)^2}+\frac{(1\cdot 3\cdot 5)^2}{3!(8\rho t)^3}+\cdots\right\},$$

which will be recognized as the usual asymptotic expansion of the function $e^{-\rho t}I_0(\rho t)$. The directness of the result, compared with usual methods, is remarkable.

We now take up the example falling in class 2, namely, the operational equation

$$h=e^{-\sqrt{\alpha p}}\ , \tag{49}$$

where α is a positive real parameter. This equation, it may be remarked, formulates the propagated voltage in a non-inductive cable.

The corresponding integral equation is

$$\frac{1}{p}\ e^{-\sqrt{\alpha p}}=\int_0^\infty h(t)e^{-pt}dt\ , \tag{50}$$

of which the solution is known. It is

$$h(t)=\frac{1}{\sqrt{\pi}}\int_0^\tau \frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\,d\tau\ , \tag{51}$$

where $\tau=4t/\alpha$.

Equation (49), it will be observed, falls in class 2; that is, it is of the form $h=\phi(\sqrt{p})$. In accordance with the Heaviside rule for this class, we expand (49) as a power series in $\sqrt{p}$. It is

$$h=1-\frac{\sqrt{\alpha p}}{1!}+\frac{\alpha p}{2!}-\frac{\alpha p\sqrt{\alpha p}}{3!}+\cdots.$$

Rearranging,

$$h=1-\left(1+\frac{\alpha p}{3!}+\frac{(\alpha p)^2}{5!}+\cdots\right)\sqrt{\alpha p}+\left(\frac{\alpha p}{2!}+\frac{(\alpha p)^2}{4!}+\cdots\right).$$

Now apply the Heaviside rule; that is, discard the series in integral powers of p, replace $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n in the first bracketed series. We get

$$(52)\quad h(t)=1-\sqrt{\frac{\alpha}{\pi t}}\left(1-\frac{1}{3\cdot 1!}\left(\frac{\alpha}{4t}\right)+\frac{1}{5\cdot 2!}\left(\frac{\alpha}{4t}\right)^2-\cdots\right),$$

which, as it is easy to verify, agrees with (51). The remarkable feature of this example is that the expansion (52) is absolutely convergent, although established by methods applicable to asymptotic expansions.

A final example will be given:

$$(53)\qquad h=\frac{1}{\sqrt{p/\beta}+1},$$

where β is a positive real parameter. This equation formulates a fairly important problem in cable telegraphy, namely the effect of a terminal resistance on the shape of the cable voltage.

The integral equation of the problem is

$$(54)\qquad \frac{1}{p\sqrt{p/\beta}+1}=\int_0^\infty h(t)e^{-pt}dt,$$

of which the solution is known. It is

$$(55)\qquad h(t)=1-e^{\beta t}\sqrt{\frac{\beta}{\pi}}\int_t^\infty\frac{e^{-\beta t}}{\sqrt{t}}\,dt.$$

Equation (53) is of class 2; expanding therefore as a power series in $\sqrt{p}$, we find

$$h=1-\sqrt{p/\beta}+p/\beta-(p/\beta)\sqrt{p/\beta}+\cdots.$$

Applying the Heaviside rule, we get

$$h(t)=1-\frac{1}{\sqrt{\pi\beta t}}\left(1-\frac{1}{2\beta t}+\frac{1.3}{(2\beta t)^2}-\cdots\right). \tag{56}$$

If the definite integral of (55) is asymptotically expanded by repeated partial integrations the resulting series is identical with (56). Furthermore, it is easy to show that it is asymptotic in the sense that the numerical error is less than the value of the last term included.

Borel has employed infinite integrals of the type

$$\int_0^\infty f(t)e^{-pt}dt$$

to sum divergent series. The foregoing suggests that they may be profitably employed to obtain asymptotic expansions of $f(t)$, when such an expansion in inverse fractional powers of t exists.

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