THE OCTOBER MEETING IN NEW YORK

The two hundred forty-fourth regular meeting of the Society was held at Columbia University, on Saturday, October 31, 1925, extending through the usual morning and afternoon sessions. The attendance included the following fifty members of the Society:

Archibald, W. L. Ayres, Babb, C. R. Ballantine, J. P. Ballantine, A. A. Bennett, R. W. Burgess, Alonzo Church, Cole, Fite, Fort, Philip Franklin, Frink, Gronwall, C. C. Grove, Hille, Dunham Jackson, Joffe, Kline, Langman, Lefschetz, Lieber, Longley, H. P. Manning, Maria, Meder, Rainich, Raynor, Reddick, R. G. D. Richardson, Ritt, Schelkunoff, Seely, Siceloff, Simons, Sosnow, M. H. Stone, Stouffer, T. Y. Thomas, Tracey, Tyler, Vallarta, Veblen, Weiss, Anna Pell Wheeler, H. S. White, Whittemore, Wiener, K. P. Williams, W. A. Wilson.

The following nine persons were elected to membership in the Society:

Mr. Ralph Palmer Agnew, Cornell University;
Mr. Julius Alsberg, consulting engineer, New York City;
Dr. Theodore Bennett, University of Illinois;
Dr. Elbert Frank Cox, West Virginia Collegiate Institute;
Mr. M. E. Hekimian, Larchmont, N.Y.;
Mr. Charles Everett Rhodes, Heidelberg University;
Mr. Homer B. Snook, Shaker Heights Schools, Cleveland;
Professor Vidal Arceo Tam, University of the Philippines;
Dr. Marian Marsh Torrey, Goucher College.

A committee consisting of Mr. S. A. Joffe and Professor Tomlinson Fort was appointed to audit the accounts of the Treasurer for the current year, and to report to the Board of Trustees at the Annual Meeting. A list of nominations of officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election.

A committee was appointed to make arrangements for the Annual Meeting. It was announced that President Birkhoff had appointed the following members to represent the Society: Professor Richard Morris at the inauguration of President

J. M. Thomas of Rutgers University on October 14, 1925; Professor J. L. Gibson at the semi-centennial of Brigham Young University on October 16, 1925.

Professor Oswald Veblen reported that through the generosity of the General Education Board, the National Academy was able to distribute funds to aid in the printing of scientific periodicals and that for the fiscal year ending July 1926, eleven hundred dollars had been granted to the BULLETIN and two thousand dollars to the TRANSACTIONS OF THE SOCIETY.

Professor J. K. Whittemore presided at the morning session, and Professor H. W. Tyler in the afternoon.

Titles and abstracts of the papers read at this meeting follow below. The papers of Dickson, Ettlinger, Franklin (second paper), Gehman, Jackson, Kellogg, Raynor, Schwatt, Stone (first paper), Thomas, Walsh, and Wilder were read by title. Mr. Kormes was introduced by Professor Pfeiffer.

1. Professor H. P. Manning: *Definitions and postulates for relativity.*

This paper attempts to give definitions and postulates for relativity, in a space of one dimension. The undefined elements are positive and negative light particles, each class forming a linear continuous series of the type of the series of all real numbers. An event consists of two light particles, one from each series. A particle (not light particle) is a series of events determined by a one-to-one correspondence of the elements taken in order of the two given series. For a particle A the local time of any one of its events is the number assigned to that event in a correspondence of them all and the series of real numbers. This number is also assigned to each of the two light particles of the event and for them called t_1 and t_2 respectively. Thus the light particles are represented by these numbers, any event by a pair of them, and a particle by an equation. Finally, time and position according to A are defined, and the relation of two systems of this kind is worked out.

2. Mr. G. Y. Rainich: *Space-time and mass.*

In order to extend to more general cases the discussion of a previous paper establishing, for the centro-symmetric case, the connection between mass and curvature, four-dimensional spaces are studied which contain an infinity of three-dimensional spaces such that their Riemann tensors are particular cases of the Riemann tensors of the containing four-dimensional space. The case is considered when the Riemann tensor determines in the three-dimensional spaces tensors of the second rank which are differentials of functions with zero divergence with a view of interpreting the residues of these functions around singularities corresponding to particles as masses of these particles.

3. Professor O. D. Kellogg: *Interpretations of Poisson's integral.*

A number of interpretations of Poisson's integral in two dimensions, such as those of Schwarz and Neumann, are well known. The present paper develops analogous results for Poisson's integral in space, and in addition gives several interpretations which appear to be new. By means of them various properties of harmonic functions of the sphere become more clearly evident and more easily proved.

4. Professor I. J. Schwatt: *The summation of a family of series of the type* $\Sigma_{k=1}^{q}(-1)^k([k/h])^p r^k$, *q either finite or infinite.*

In addition to the form given in the title of the paper, the value of summations like $\Sigma_{k=1}^{q}(-1)^k([k/h])^p \sin kxr^k$, $[k/h]$ being the largest integer in k/h, and similar forms are considered. The work involves several principles in the operation with series.

5. Professor A. A. Bennett: *Large primes have at least five consecutive quadratic residues.*

The primes are distributed among a complete system of nonoverlapping linear forms, and it is shown that for each prime greater than 149 there are at least five consecutive quadratic residues by actually exhibiting for each of them such a set. The discussion provides therefore something more than a mere existence proof. The method is elementary, but would probably involve an unreasonable amount of labor were it applied to still higher cases.

6. Mr. Mark Kormes: *A note on the functional equation* $f(x+y)=f(x)+f(y)$.

In this paper the following theorem is proved. Every solution of the functional equation $f(x+y)=f(x)+f(y)$ which is bounded on a set M of measure $m(M)>0$ has the form $A \cdot x$.

7. Dr. H. M. Gehman: *Some theorems on continuous curves containing no simple closed curve.*

The author proves the following theorems: (I) If $T+T'$ is a closed, totally disconnected subset of a bounded continuum M, and if T is the set of all cut points of M, then M is a continuous curve containing no simple closed curve and T is the set of all end points of M. (II) If T is a closed, totally disconnected subset of a bounded continuum M, then M is a continuous curve containing no simple closed curve, with all its end points in T, if and only if no proper connected subset of M contains T. (III) A bounded continuum M is a continuous curve, if and only if every closed, totally disconnected subset T of M is a subset of some subcontinuum of M which contains no proper connected subset containing T. (IV) A closed subset M of a continuous curve S is a subset of a continuous curve containing no simple closed curve and lying in S, if and only if every maximal connected subset of M is a continuous curve containing no simple closed curve, and not more than a finite number of them are of diameter greater than any given positive number. (V) Every closed, bounded, totally disconnected point set is identical with the end points of some continuous curve containing no simple closed curve.

8. Dr. H. M. Gehman: *On irredundant sets of postulates.*

This paper will appear in full in an early issue of this BULLETIN.

9. Professor R. L. Wilder: *A property which characterizes continuous curves.*

Let M be a point set, C_1 and C_2 mutually exclusive closed subsets of M, and K a connected subset of M such that (1) K contains no points of the set C_1+C_2 and (2) both C_1 and C_2 contain limit points of K. Then K, together with its limit points in C_1+C_2, is called a set $K(C_1, C_2)M$. If M has the property that for every two of its mutually exclusive closed subsets C_1, C_2 and every connected set N containing points of both C_1 and C_2, there exists some set $K(C_1, C_2)M$ which has a point in common with N, then M is called normally connected. The following theorem is proved: *In order that a continuum M should be a continuous curve, it is necessary and sufficient that it be normally connected.*

10. Professor R. L. Wilder: *A theorem on connected point sets which are connected im kleinen.*

The following theorem is established: Let K and N be two mutually exclusive point sets, and M a connected, connected im kleinen point set which has at least one point in common with each of the sets K and N. Then there exists a point set H, a subset of M, such that H is connected and contains no point of either K or N, but such that K and N each contain at least one point of M which is a limit point of H.

11. Professor H. J. Ettlinger: *Note on the continuity of a function defined by a definite Lebesgue integral.*

The author shows that the following theorem recently proved by R. L. Jeffery (AMERICAN MATHEMATICAL MONTHLY, vol. 32 (1925), pp. 297-

299) can be obtained as a direct consequence of the Duhamel-Moore theorem: If (1) $f(x, y)$ is a real bounded function of (x, y) on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$, and (2) $f(x, y)$ is a summable function of x on (a, b) for each y in (a, b), and (3) $f(x, y)$ is a continuous function of y at $\bar{y}$, for every x in (a, b), except a null set M_0, then $F(y) = L\int_a^b f(x, y)\,dx$ is continuous at every point $\bar{y}$ in (a, b). The same result can also be obtained by the use of Lebesgue's theorem to the effect that the limit of a sequence of bounded summable functions is a summable function and the integral of the limit is the limit of the integral, where a null set of the interval may be disregarded. The above theorem may also be stated for m variables.

12. Professor J. L. Walsh: *On the expansion of an analytic function in a series of polynomials.*

This paper proves the following theorem, by means of results on conformal mapping due to Courant (GÖTTINGER NACHRICHTEN, 1914, pp. 101-109): If $f(z)$ is an analytic function of z in the interior of a Jordan curve C and if $f(z)$ is continuous in the closed region consisting of C and its interior, then throughout that closed region $f(z)$ can be expanded in a series of polynomials in z, the series converging uniformly in the same closed region.

13. Professor J. L. Walsh: *On the position of the roots of entire functions of genus zero and unity.*

This paper extends to entire function of genus zero and unity results formerly proved by the author for polynomials.

14. Professor G. E. Raynor: *On isolated singular points of harmonic functions.*

In the BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, vol. 52 (1924), Picard has given a characteristic of a harmonic function in the neighborhood of an isolated singular point in which neighborhood the function tends towards plus infinity. The purpose of the present paper is to characterize any harmonic function in the neighborhood of an isolated singular point and to obtain Picard's result as a special case and by a method different from his.

15. Professor W. A. Wilson: *On the structure of a limited continuum, irreducible between two points.*

Let ab denote the continuum, x one of its points, and X the oscillatory set about x. If neither of the saturated semi-continua X_a and X_b contained in $ab - X$ and containing a and b, respectively, is void, X is called *complete* if x is a common limiting point of X_a and X_b. (Likewise, A is complete of A_b is not void and a is a limiting point of A_b.) The necessary and sufficient condition for ab to be the union of an aggregate of complete oscillatory sets without common points is that it contain no indecomposable continuum which is not a continuum of condensation. If $K=\{X\}$ denotes the aggregate of complete oscillatory sets of such a continuum, K has the order type of a linear segment $T=(0\leqq t\leqq 1)$. Furthermore the function $X=f(t)$ is an "upper semi-continuous function," i. e., $X=f(t)$ is the upper closed limit of $X'=f(t')$ as $t'\rightarrow t$ at each point of T.

16. Dr. T. H. Gronwall: *The algebraic structure of the formulas in plane trigonometry.* Third paper.

The present paper contains the complete solution of the derived equations of Hilbert, and the determination of the characteristic function, for the formulas of plane trigonometry.

17. Dr. T. H. Gronwall: *Summation of series and conformal mapping.*

Let $z=f(w)$ map $|w|<1$ on a simple region D in the z-plane, where D is interior to $|z|<1$, the boundary of D consists of a finite number of analytic curves, and $z=0$ for $w=0$, $z=1$ for $w=1$, and

$$w-1=c(z-1)^k+\cdots,$$

with $k>1$, at $z=1$. Moreover, let $g(w)$ be holomorphic and different from zero for $|w|<1$ and $g(w)\rightarrow 0$ as $w\rightarrow 1$ in an angle $<\pi$ with vertex at $w=1$ and interior to $|w|<1$, and let all coefficients in the expansion $1/g(w)=\Sigma c_n w^n$ be positive. The "sum" of the series

$$u_0+u_1+\cdots+u_n+\cdots$$

is now defined as $\lim_{n\rightarrow\infty} U_n$ (when existing), where U_n is defined by the formal identity

$$\Sigma_0^\infty u_n z^n=g(w)\ \Sigma_0^\infty c_n U_n w^n.$$

It is shown that when the series is summable (U) with the sum s, then $\Sigma u_n z^n\rightarrow s$ as $z\rightarrow 1$ in the angle referred to above. Under some additional restrictions on the order of magnitude of $g(w)$ when approaching a singularity on the unit circle, it is shown that when $u_0+u_1+\cdots+u_n$ is summable by Cesàro's means of *any* order, then it is summable (U) with the same sum; moreover, a very general comparison theorem for sums (U) with different values of $f(w)$ and $g(w)$ is established. The summation method of de la Vallée Poussin is the special case of (U) where

$$w=4z/(1+z)^2,\ g(w)=(1-w)^{1/2}.$$

18. Dr. T. H. Gronwall: *A new form of the remainder in the binomial series, with applications.*

Writing the binomial expansion of $(1-x)^{-k}$ with remainder term in the form

$$(1-x)^{-k}=\ \Sigma_0^{n-1}c_\nu x^\nu+x^n(1-x)^{-k}f(x),$$

where $k>0$, it is shown that for $0<x<1$, $f(x)$ is monotone increasing for $k<1$, constant $(=1)$ for $k=1$, decreasing for $k>1$. When $0<k\leqq 1$, the coefficients in the power series for $f(x)$ are all positive, and

$$\left|(1-x)^{-k}-\Sigma_0^{n-1}c_\nu x^\nu\right|<|x|^n|1-x|^{-k}$$

for $|x|\leqq 1$ but $x\neq 1$. These formulas are very useful in the derivation of various asymptotic expansions.

19. Professor Philip Franklin: *Almost-periodic functions of two variables.*

In this paper we generalize the definition of almost-periodic functions of one variable, as given by Harold Bohr, to the case of two variables, and extend many of his results. We also obtain a few properties characteristic of the two variable case. As an example, we have the theorem that any almost-periodic function of two variables is, along any straight line, an almost-periodic function of the distance along that line.

20. Professor Philip Franklin: *The elementary character of certain integrals related to figures bounded by spheres and planes.*

In this note we show that the problem of calculating any surface, volume, or center of gravity of a surface or volume of a figure formed entirely from spheres and planes leads to an elementary integral, the result being an elementary function (combination of algebraic, trigonometric, and inverse trigonometric) of the parameters determining the figure.

21. Mr. W. L. Ayres: *Concerning the arcs and domains of a continuous curve.*

It is proved in this paper that if M and N are continuous curves and N is a subset of M, then, for any positive ϵ, $M-N$ contains not more than a finite number of maximal domains with respect to M of diameter greater than ϵ. It is shown that a continuous curve M cannot contain, for any positive ϵ, more than a finite number of arcs, having no common points except possibly end points and of diameter greater than ϵ, such that no point of any arc α, except its end points, is a limit point of $M-\alpha$. Also two theorems proved by R. L. Wilder are generalized.

22. Professor Norbert Wiener: *A new method in periodogram analysis.*

The author develops the theory of a new method of periodogram analysis, connected with the correlation-coefficient method of G. I. Taylor for the analysis and description of irregular sequences of data.

23. Dr. M. H. Stone: *The convergence of Bessel's series.*

An extension of Birkhoff's method of discussing expansion problems is here applied to Bessel's series. The leading results of W. H. Young (PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, 1920) are thus demonstrated by an independent treatment for developments in terms of the functions $J_\nu(\lambda_k x)$ where $\nu > -\frac{1}{2}$, $J_\nu(\lambda_k) = 0$.

24. Dr. M. H. Stone: *The Borel summability of Fourier series.*

The Borel summability of Fourier series is discussed by means of an integral of the form $\int_0^A e^{-\alpha} f(x, \alpha) d\alpha$, in which $A \to \infty$. C. N. Moore has announced results showing the comparative ineffectiveness of this sum. On introducing a convergence factor, the integral becomes $\int_0^A (1-\alpha/A)e^{-\alpha} f(x,\alpha)d\alpha$. It is shown that as $A \to \infty$ the integral converges for almost all values of x to the function for which the series is defined. It is found possible to obtain similar results by applying a modification of the Borel sum to the Green's function for the differential system $u' + \lambda u = 0$, $u(0) - u(1) = 0$.

25. Dr. T. Y. Thomas: *The identities of affinely connected manifolds.*

In this paper the idea of a complete set of identities of an invariant is introduced as the set of algebraically independent identities furnishing all the algebraic conditions on the invariant. Hence every identity satisfied by the invariant can be deduced by algebraic processes from the identities of the complete set. The paper gives the complete set of identities of the affine and metric normal tensors which constitute complete sets of invariants of general affinely connected manifolds and Riemann spaces respectively. In addition the completeness proof for the identities satisfied by the curvature tensor $B^i_{\alpha\beta\gamma}$ in the general manifold, and for the covariant form of the curvature tensor $B_{\alpha\beta\gamma\delta}$ in the Riemann geometry is given.

26. Professor M. J. Babb: *The research manuscripts and library of Dr. Robert Adrain, professor of mathematics at Rutgers, Columbia, and Pennsylvania.* Preliminary report.

In this paper, a brief description is given of the manuscripts, books, and portrait of Robert Adrain recently presented to the University of Pennsylvania by his great-grandchildren through Frederick S. Duncan, Esq., of Englewood, N. J., together with the records and correspondence that have recently come to light at that University.

27. Professor Dunham Jackson: *Note on the convergence of Fourier series.*

In a recent number of the AMERICAN MATHEMATICAL MONTHLY (December, 1924), Franklin has given a very clear and simple proof of the convergence of Fourier series under hypotheses sufficiently general to cover the most important applications. The purpose of this note, starting with the same hypotheses, is to present a proof which is believed to be materially simpler still.

28. Professor L. E. Dickson: *New division algebras.*

Only one type of division algebras is known. This paper obtains an infinitude of new types, one for each solvable group. The paper has been offered for publication in the TRANSACTIONS.

29. Dr. H. M. Gehman: *A theorem on continuous curves in space of n dimensions.*

The author has proved that any continuous curve containing no simple closed curve and lying in a space of n dimensions is in continuous (1, 1) correspondence with some continuous curve of the same type in space of two dimensions. This answers a question proposed by S. Mazurkiewicz (FUNDAMENTA MATHEMATICAE, vol. 2 (1921), p. 130).

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Secretary.