RESOLVENT SEXTICS OF QUINTIC EQUATIONS

BY L. E. DICKSON

1. *Introduction.* The object of this paper is to give simple derivations of the classic resolvents which have been obtained heretofore by elaborate computations.

Jacobi* established the form of a remarkable resolvent, but neither found the values of the coefficients nor gave the simple details (§ 2 below) which lead directly to that form.

Cayley† was not aware of Jacobi's work when he fully computed the same resolvent. Noting that its roots are functions of the differences of the roots x_i of the quintic, he first computed at length the resolvent sextic under the restriction that $x_5 = 0$. Then the coefficients were "completed by the introduction of the terms involving the constant coefficient of the quintic." No details were given of the latter long computation, which may perhaps be best made by utilizing the fact that the coefficients are seminvariants. The simple new method employed here (§ 3) makes initial use of the latter fact as well as of a lemma which reduces the search for the needed seminvariants of the quintic to a mere inspection of the invariants of a quartic.

From the Jacobi-Cayley resolvent (which is a simple transform of the old Malfatti resolvent) it is an immediate step (§ 5) to the noteworthy covariant resolvent discovered by Perrin,‡ and independently by McClintock,§ each time as the final step of a long computation.

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* JOURNAL FÜR MATHEMATIK, vol. 13 (1835), pp. 340–52; WERKE, vol. 3, 1884, pp. 269–84.

† PHILOSOPHICAL TRANSACTIONS, London, vol. 151 (1861), pp. 263–76; COLLECTED MATHEMATICAL PAPERS, vol. 4, pp. 309–24.

‡ COMPTES RENDUS DU DEUXIÈME CONGRÈS INTERNATIONAL DES MATHÉMATICIENS, Paris, 1902, pp. 199–223. Announced in BULLETIN DE LA SOCIÉTÉ DE FRANCE, vol. 11 (1882-83), pp. 64–65.

§ AMERICAN JOURNAL, vol. 8 (1886), pp. 45–84; vol. 20 (1898), pp. 157–192.

2. *The Symmetric Functions of* $z_1, \cdots, z_6$. Writing ij for $x_i x_j$, we consider the function

$$12345 = 12 + 23 + 34 + 45 + 51.$$

It is unaltered by the substitutions $a = (12345)$, $b = (25)(34)$. Since $b^{-1}ab = a^{-1}$, a and b generate a group of ten even substitutions. Since 12345 is therefore unaltered by these ten, it takes at most 60/10 distinct values under the group G of all 60 even substitutions. It actually takes the six distinct values given by the first (positive) parts of

$$z_1 = 12345 - 13524, \quad z_2 = 12453 - 14325,$$
$$z_3 = 12534 - 15423, \quad z_4 = 15243 - 12354,$$
$$z_5 = 14235 - 12543, \quad z_6 = 13254 - 12435.$$

In fact, (345), (354), (253), (243), (23)(45) replace z_1 by $z_2, \cdots, z_6$. Hence $z_1, \cdots, z_6$ are merely permuted by each of the 60 even substitutions. Next, every odd substitution O replaces each z_i by the negative of some z. For, (2354) replaces z_1 by $-z_1$. Let E be one of the even substitutions which replaces z_i by z_1 and write E_1 for the even substitution $(2354)^{-1}E^{-1}$. Then $O = E(2354)E_1$ replaces z_i by the function by which E_1 replaces $-z_1$ and that function is the negative of some z.

Hence any homogeneous symmetric function of $z_1, \cdots, z_6$ of even degree is symmetric in $x_1, \cdots, x_5$. But if it is of odd degree in the z's, it merely changes sign when any two x's are interchanged and hence is divisible by the product of the ten differences of the x's, the quotient being symmetric in the x's.

3. *The Jacobi-Cayley Resolvent.* The discriminant Δ of

$$f(x,y) = a_0x^5 + 5a_1x^4y + 10a_2x^3y^2 + 10a_3x^2y^3 + 5a_4xy^4 + a_5y^5$$

is defined to be the polynomial such that $5^5 a_0^{-8}\Delta$ is equal to the product of the squares of the ten differences of the roots x_i of $f(x, 1) = 0$.

In the sextic having the roots $z_1, \cdots, z_6$, the coefficients of z^5 and z^3 are zero by § 2, being of odd degrees 1 and 3 in the z's, so that their degrees in the x's are less than the degree 10 of the product Π of the differences of the x's. The coefficient of z is the product of a numerical constant by Π or by $a_0^{-4}\sqrt{\Delta}$. It is convenient to multiply the sextic by a_0^6. We get

$$a_0^6 z^6 + a_0^4 u_1 z^4 + a_0^2 u_2 z^2 + u_3 = \nu a_0^2 \sqrt{\Delta} z,$$

where ν is a numerical constant, while $a_0^{-2i} u_i$ is the sum of the products of the z's taken $2i$ at a time and hence is of total degree $4i$ in the x's and of degree $2i$ in any one root x. By § 2, it is symmetric in the x's. It is expressible as a polynomial in the differences of the x's, since

$$z_1 = (1-5)(2-5) + (2-5)(3-5) + (3-5)(4-5) \\ -(2-5)(4-5) - (4-5)(1-5) - (1-5)(3-5).$$

It follows* that u_i is a seminvariant of f of degree $2i$ and weight $4i$. By a seminvariant S of f is meant a homogeneous isobaric polynomial in $a_0, \cdots, a_5$ for which $\Omega S = 0$, i. e., is annihilated by the operator

$$\Omega = a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + 3a_2 \frac{\partial}{\partial a_3} + 4a_3 \frac{\partial}{\partial a_4} + 5a_4 \frac{\partial}{\partial a_5}.$$

Since u_1 is of degree 2 and weight 4, it lacks a_5 and is the product of I by a numerical constant, as shown by the following lemma.

LEMMA. *If a seminvariant S of the quintic $f(x, y)$ lacks a_5, it is a seminvariant of the quartic*

$$q = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4.$$

If the weight of S is double its degree, it is an invariant of q and hence is a polynomial in

$$I = a_0 a_4 - 4a_1 a_3 + 3a_2^2,$$
$$J = a_0 a_2 a_4 - a_0 a_3^2 + 2a_1 a_2 a_3 - a_1^2 a_4 - a_2^3.$$

For, S is homogeneous and isobaric and is annihilated by the operator derived from Ω by suppressing the final

* Dickson, *Algebraic Invariants*, New York, 1914, p. 53.

derivative. Hence S is a seminvariant of q. A seminvariant of degree d and weight w of a binary form of order p is the leader of a unique covariant of order $pd-2w$ (*Invariants*, p. 43 and Ex. 1, p. 40). It is therefore an invariant if $p=4$, $w=2d$.

A known seminvariant of the same degree 4 and weight 8 as u_2 is

$$T = a_0^2a_3a_5 - 3a_0a_1a_2a_5 - 5a_0a_1a_3a_4 + 10a_0a_2^2a_4 - 4a_0a_2a_3^2 + 2a_1^3a_5 - 5a_1^2a_2a_4 + 14a_1^2a_3^2 - 16a_1a_2^2a_3 + 6a_2^4$$

(it suffices to verify that $\Omega T=0$). We delete the term $a_0^2a_3a_5$ from u_2 by subtracting a multiple of T. In the resulting seminvariant v, the only terms involving a_5 are those in $a_5(\alpha a_0a_1a_2+\beta a_1^3)$, since the terms in parenthesis, together with the deleted term $a_0^2a_3$, are the only possible terms of degree 3 and weight 3. By inspection, Ωv is the sum of

$$\alpha a_0^2a_2a_5 + (2\alpha+3\beta)a_0a_1^2a_5$$

and terms free of a_5. Hence $\alpha=\beta=0$. Thus v lacks a_5 and by the Lemma is an invariant of q and hence is a product of I^2 by a constant. Thus u_2 is a linear combination of I^2 and T.

To determine u_3 we shall employ the seminvariant P of the same degree 6 and same weight 12 (cf. § 6):

$$\begin{aligned} P = {} & a_0^2a_2a_5^2 - 2a_0^2a_3a_4a_5 + a_0^2a_4^3 - a_0a_1^2a_5^2 - 4a_0a_1a_2a_4a_5 \\ & + 8a_0a_1a_3^2a_5 - 2a_0a_1a_3a_4^2 - 2a_0a_2^2a_3a_5 + 14a_0a_2^2a_4^2 \\ & - 22a_0a_2a_3^2a_4 + 9a_0a_3^4 + 6a_1^3a_4a_5 - 12a_1^2a_2a_3a_5 - 15a_1^2a_2a_4^2 \\ & + 10a_1^2a_3^2a_4 + 6a_1a_2^3a_5 + 30a_1a_2^2a_3a_4 - 20a_1a_2a_3^3 \\ & - 15a_2^4a_4 + 10a_2^3a_3^2. \end{aligned}$$

We use a_0P and TI to delete the terms $a_0^3a_2a_5^2$ and $a_0^3a_3a_4a_5$ from u_3. In the resulting seminvariant S, the terms having the factor a_5 are those in

$$\begin{aligned} & \alpha a_0^2a_1^2a_5^2 + \beta a_0^2a_1a_2a_4a_5 + \gamma a_0^2a_1a_3^2a_5 + a a_0^2a_2^2a_3a_5 \\ & + b a_0a_1^3a_4a_5 + c a_0a_1^2a_2a_3a_5 + d a_0a_1a_2^3a_5 + e a_1^4a_3a_5 + g a_1^3a_2^2a_5. \end{aligned}$$

These terms alone furnish the part of ΩS involving a_5:

$$2\alpha a_0^3 a_1 a_5^2 + \beta a_0^3 a_2 a_4 a_5 + \gamma a_0^3 a_3^2 a_5 + (3b + 2\beta + 10\alpha) a_0^2 a_1^2 a_4 a_5 + (2c + 4a + 4\beta + 6\gamma) a_0^2 a_1 a_2 a_3 a_5 + (d + 3a) a_0^2 a_2^3 a_5 + (2c + 4b + 4e) a_0 a_1^3 a_3 a_5 + (3c + 3g + 6d) a_0 a_1^2 a_2^2 a_5 + (4g + 3e) a_1^4 a_2 a_5 .$$

The conditions that this be zero identically require that α, β, γ, a, b, c, d, e, g shall all vanish. Since S therefore lacks a_5, the lemma shows that it is a sum of terms $I^r J^s$ whose degree is $2r + 3s = 6$, whence either $s = 0$, $r = 3$ or $s = 2$, $r = 0$. Thus u_3 is a linear combination of I^3, J^2; $a_0 P$, TI. Hence the sextic is of the form

$$a_0^6 z^6 + a_0^4 \alpha I z^4 + a_0^2(\beta I^2 + \gamma T) z^2 + \delta I^3 + \epsilon J^2 + \lambda a_0 P + \mu TI = a_0^2 \nu \sqrt{\Delta} z,$$

where the Greek letters are numerical constants. To find their values, we employ the special quintic having $x_3 = -x_1$, $x_4 = -x_2$, $x_5 = 0$:

$$x(x^2 - x_1^2)(x^2 - x_2^2) = x^5 + 10 a_2 x^3 + 5 a_4 x = 0,$$

where $10 a_2 = -x_1^2 - x_2^2$, $5 a_4 = x_1^2 x_2^2$. Then

$$z_1 = z_3 = (x_1 + x_2)^2, \qquad z_2 = z_4 = -(x_1 - x_2)^2,$$
$$z_5 = x_1^2 - x_2^2 - 4x_1 x_2, \qquad z_6 = x_2^2 - x_1^2 - 4 x_1 x_2 .$$

Using temporarily the abbreviations

$$s = x_1^2 + x_2^2, \qquad p = x_1^2 x_2^2, \qquad t = x_1 x_2,$$

we see that z_1 and z_2 are the roots of $z^2 - 4tz + 4p - s^2 = 0$, while z_5 and z_6 are the roots of $z^2 + 8tz + 20p - s^2 = 0$. Hence the sextic is

$$z^6 - (20p + 3s^2) z^4 + (240 p^2 - 8ps^2 + 3s^4) z^2 + (4p - s^2)^2 (20p - s^2) = 128 tp(4p - s^2) z.$$

Replacing s, p, t by their values in terms of a_2, a_4, we get

$$z^6 - 100(a_4 + 3a_2^2) z^4 + 2000(3a_4^2 - 2a_4 a_2^2 + 15 a_2^4) + 40000(a_4^3 - 11 a_4^2 a_2^2 + 35 a_4 a_2^4 - 25 a_2^6) = 12800 a_4 (a_4 - 5 a_2^2) \sqrt{5 a_4} z.$$

But for $a_0=1$, $a_1=a_3=a_5=0$, we have

$$I=a_4+3a_2^2,\qquad J=a_2a_4-a_2^3,\qquad T=10a_2^2a_4+6a_2^4,$$
$$\Delta=16^2a_4^3(a_4-5a_2^2)^2,\qquad P=a_4^3+14a_2^2a_4^2-15a_2^4a_4.$$

We see at once that $\alpha=-100$, $\beta=6000$, $\gamma=-4000$, $\nu=800\sqrt{5}$. By the terms free of z,

$$\delta+\lambda=40000,\qquad 9\delta+\epsilon+14\lambda+10\mu=-11\cdot 40000,$$
$$27\delta-2\epsilon-15\lambda+36\mu=35\cdot 40000,$$
$$27\delta+\epsilon+18\mu=-25\cdot 40000.$$

Hence $\lambda=40000$, $\epsilon=-25\lambda$, $\mu=\delta=0$. The resolvent sextic is therefore

$$a_0^6z^6-100a_0^4Iz^4+2000a_0^2(3I^2-2T)z^2-800\sqrt{5}\,a_0^2\sqrt{\Delta}z$$
$$+40000(a_0P-25J^2)=0.$$

4. *Canonizant C.* The covariant of f,

$$C=Jx^3+J_1x^2y+J_2xy^2+J_3y^3,$$

having the leader J, may readily be found by means of the annihilator (*Invariants*, p. 39)

$$O=5a_1\frac{\partial}{\partial a_0}+4a_2\frac{\partial}{\partial a_1}+3a_3\frac{\partial}{\partial a_2}+2a_4\frac{\partial}{\partial a_3}+a_5\frac{\partial}{\partial a_4}.$$

We get

$$OJ=J_1=\begin{vmatrix} a_0 & a_1 & a_3\\ a_1 & a_2 & a_4\\ a_2 & a_3 & a_5\end{vmatrix},\qquad \tfrac{1}{2}OJ_1=J_2=\begin{vmatrix} a_0 & a_2 & a_3\\ a_1 & a_3 & a_4\\ a_2 & a_4 & a_5\end{vmatrix},$$

$$\tfrac{1}{3}OJ_2=J_3=\begin{vmatrix} a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix},\qquad J=\begin{vmatrix} a_0 & a_1 & a_2\\ a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\end{vmatrix}.$$

Hence

$$C=\begin{vmatrix} a_0x+a_1y & a_1x+a_2y & a_2x+a_3y\\ a_1x+a_2y & a_2x+a_3y & a_3x+a_4y\\ a_2x+a_3y & a_3x+a_4y & a_4x+a_5y\end{vmatrix}.$$

The name canonizant is given to C since its three linear factors u, v, w have the property* that $f = u^5 + v^5 + w^5$. For a direct derivation of C, see § 6. An elementary verification that C is a covariant may be made by manipulating a single determinant. Under the transformation $x = X + \varepsilon Y$, $y = Y$, f becomes $F = A_0 X^5 + 5A_1 X^4 Y + \cdots$, where†

$$
\begin{aligned}
A_0 &= a_0, \\
A_1 &= a_1 + \varepsilon a_0, \\
A_2 &= a_2 + 2a_1\varepsilon + a_0\varepsilon^2, \\
A_3 &= a_3 + 3a_2\varepsilon + 3a_1\varepsilon^2 + a_0\varepsilon^3, \\
A_4 &= a_4 + 4a_3\varepsilon + 6a_2\varepsilon^2 + 4a_1\varepsilon^3 + a_0\varepsilon^4, \\
A_5 &= a_5 + 5a_4\varepsilon + 10a_3\varepsilon^2 + 10a_2\varepsilon^3 + 5a_1\varepsilon^4 + a_0\varepsilon^5.
\end{aligned}
$$

To the elements of the second row of C add the products of those of the first row by ε. To the elements of the third row add the products of those of the first row by ε^2 and those of the original second row by 2ε. Replace x and y by their values. To the new determinant apply the corresponding operations on columns instead of rows. We get a determinant of type C written in capital letters. Finally, the interchange of x with y and hence of a_0 with a_5, a_1 with a_4, a_2 with a_3, replaces C by a determinant which reduces to C by writing its rows in reverse order and then the columns in reverse order.

5. *Covariant Resolvent.* We employ the linear covariant $L = Px + Qy$, where P was defined in § 3 and Q is derived from P by the substitution $(a_0 a_5)(a_1 a_4)(a_2 a_3)$ induced by the interchange of x and y. The constant term $40000(a_0 P - 25J^2)$ of the resolvent in § 3 is therefore the leader of the covariant $K = 40000\,(fL - 25C^2)$ of order 6. Equated to zero, it gives the covariant resolvent of Perrin, which was rediscovered by McClintock and called the central resolvent.

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* Salmon, *Modern Higher Algebra*, 4th ed., p. 153; German translation by Fiedler, p. 199.

† We may write symbolically a^i for a_i, $f = (x + a_1 y)^5$. Then $F = (X + \beta Y)^5$, $\beta = a_1 + \varepsilon$. After expansion, the terms free of a_1 are to be multiplied by a_0. From β^i we get A_i.

For its use as a resolvent of the quintic $f(x, 1) = 0$, it is essential to know the expressions for its roots in terms of $x_1, \cdots, x_5$. To find these expressions, we shall give a process indicated without proof by McClintock, but requiring correction by inserting factors a_0. Write

$$\Phi_1 = a_0 z_1 = a_0 \sum x_1(x_2 - x_3),$$

where (and below) each of the five terms of $\sum$ are derived from the preceding term by the substitution (12345). Replacing each x_i by its reciprocal, we get

$$a_0 \sum \frac{(x_3 - x_2)x_4 x_5}{x_1 x_2 x_3 x_4 x_5} = -\frac{a_0}{a_5} \Psi_1, \quad \Psi_1 = a_0 \sum (x_3 - x_2) x_4 x_5.$$

Similarly, from Ψ_1 we get $-\Phi_1 a_0/a_5$. We shall prove that

$$G = \prod_{i=1}^{6} (\Phi_i x - \Psi_i y)$$

is a covariant of f, where $\Phi_i = a_0 z_i$, and Ψ_i is derived from Φ_i by replacing each root x by its reciprocal. Since the leader of G is $\Phi_1 \cdots \Phi_6$, which is equal to the constant term of the resolvent in § 3, and hence to the leader of K, it will follow that $G = K$.

Apply transformation $x = Y$, $y = X$ to $f = a_0 \prod (x - x_1 y)$. We get $F = a_5 \prod (X - x_1^{-1} Y)$. The function Φ_1 for F is

$$a_5 \sum x_1^{-1}(x_2^{-1} - x_3^{-1}) = \frac{a_5}{a_0}\left(-\frac{a_0}{a_5}\Psi_1\right) = -\Psi_1.$$

The function Ψ_1 for F is the product of a_5/a_0 by the function $-\Phi_1 a_0/a_5$ obtained above from Ψ_1 by replacing each x_i by its reciprocal. Hence G for F is

$$\prod (-\Psi_1 X + \Phi_1 Y) = \prod (\Phi_1 x - \Psi_1 y) = G.$$

Next, apply transformation $x = X + tY$, $y = Y$ to f. We get $F = a_0 \prod (X - X_1 Y)$, where $X_1 = x_1 - t$. The function Φ_1 for F is the seminvariant Φ_1 itself. The function Ψ_1 for F is

$$a_0 \sum (x_3 - x_2)(x_4 - t)(x_5 - t) = \Psi_1 - t\Phi_1 \equiv M_1.$$

Hence G for F is

$$\prod(\Phi_1 X - M_1 Y) = \prod[\Phi_1 x - (t\Phi_1 + M_1)y] = G.$$

Thus *the covariant resolvent* $K(x,1) = 0$ *has the roots* Ψ_i/Φ_i.

A like process enables us to write down at once the linear factors of a covariant of order n whose leader is a seminvariant which is the product of n rational functions of the x's.

6. *Another Derivation of* C *and* L. If in a covariant φ of f we replace $x^r y^s$ by $(-1)^s \partial^{r+s}/(\partial y^r \partial x^s)$, i. e. replace the products of powers of x and y by symbolic products of powers of $\partial/\partial y$ and $-\partial/\partial x$, and apply the resulting operator to another covariant ψ of f, we obtain a covariant $[\varphi, \psi]$ of f (*Invariants*, top p. 61).

The quintic f has the covariant*

$$i = Ix^2 + I_1 xy + I_2 y^2, \quad I_1 = OI = a_0 a_5 - 3a_1 a_4 + 2a_2 a_3,$$
$$I_2 = \tfrac{1}{2} O I_1 = a_1 a_5 - 4a_2 a_4 + 3a_3^2.$$

Then

$$-\tfrac{1}{60}[i, f] = C, \quad -\tfrac{1}{2}[i, C] = L = Px + Qy.$$

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* It is the invariant I of the fourth polar of f.

AN ACKNOWLEDGEMENT

BY J. H. M. WEDDERBURN

Professor G. Scorza has kindly called my attention to the fact that the result of my note entitled *A theorem on simple algebras* (this BULLETIN, vol. 31, pp. 11–13) was given by him in his book, *Corpi Numerici e Algebre* (1921), pp. 346–352. I regret that I was unaware of this at the time the paper was published, and I take this means of acknowledging Professor Scorza's priority.