ON THE POLYNOMIAL OF THE BEST APPROXIMATION TO A GIVEN CONTINUOUS FUNCTION*

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1. *A Theorem on Minimizing Polynomials.* Let $f(x)$ and $p(x)$ be defined on a finite interval (a, b); $f(x)$ is bounded and integrable, $p(x)$ is integrable and not negative.

THEOREM I.† *If there exist two numbers α, β such that $a \leqq \alpha < \beta \leqq b$, and such that*

$$\int_c^d p(x)\, dx > 0$$

whenever $\alpha \leqq c < d \leqq \beta$, then there exists one and only one polynomial of degree $\leqq n$ minimizing the integral

$$I_{nk} = \int_a^b p(x)\, |f(x) - U_{nk}(x)|^k\, dx,$$

where

$$U_{nk}(x) = \sum_{i=0}^{n} u_{ik} x^i,$$

provided that $k > 1$. If $k = 1$, the proof of existence applies without change; and the approximating polynomial is unique, if $f(x)$ is continuous on (a, b), and if

$$\int_c^d p(x)\, dx > 0$$

whenever $a \leqq c < d \leqq b$.

The proof may be organized as follows.

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† Cf. G. Pólya, *Sur un algorithme ...*, COMPTES RENDUS, vol. 157 (1913), pp. 840-843; D. Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117-128, *Note on a class of polynomials of approximation*, ibid., vol. 22 (1921), pp. 320-326, *A generalized problem in weighted approximation*, ibid., vol. 26 (1924), pp. 133-154, *Note on the convergence of weighted trigonometric series*, this BULLETIN, vol. 29 (1923), pp. 259-263.

2. *Existence of a Minimizing Polynomial.* The value of n being regarded as fixed, take

$$\delta = \frac{\beta - \alpha}{8n^2}. \tag{1}$$

For $y \pm \delta$ belonging to (α, β), we have

$$\int_y^{y+\delta} p(x)\,dx, \quad \int_{y-\delta}^{y} p(x)\,dx \geqq h(\delta) = h_n, \tag{2}$$

since these integrals are continuous and positive functions of y; h_n depends on n only. For an arbitrarily large $A > 0$ we can find, according to Kirchberger,* certain K_i such that *any one* of the inequalities $|u_{ik}| > K_i$ $(i = 0, 1, 2, \cdots, n)$ implies

$$\begin{aligned} &\max |U_{nk}(x)| \text{ in } (\alpha, \beta) = |U_{nk}(\xi)| = M > 2\,\Phi + 2\sqrt[k]{A/h_n}, \\ &\Phi = \max |f(x)| \text{ in } (a, b), \qquad \alpha \leqq \xi \leqq \beta. \end{aligned} \tag{3}$$

Using Markoff's theorem† and (1), we find, for $|x - \xi| \leqq \delta$,

$$|U_{nk}(x) - U_{nk}(\xi)| \leqq |x - \xi| \cdot \frac{2n^2 M}{\beta - \alpha} \leqq \frac{M}{4},$$

$$|U_n(x)| > \frac{M}{2} > \Phi + \sqrt[k]{A/h_n},$$

$$|f(x) - U_{nk}(x)| > \sqrt[k]{A/h_n}, \tag{4}$$

whence it follows that $I_{nk} > A$, since at least one of the intervals $(\xi - \delta, \xi)$, $(\xi, \xi + \delta)$ belongs to (α, β). Therefore we must take $|u_{ik}| \leqq K_i$ $(i = 0, 1, 2, \cdots, n)$, which proves the existence of a minimizing polynomial. We use the notation

$$M_{nk} = \min I_{nk} = \int_a^b p(x)\,|f(x) - P_{nk}(x)|^k\,dx. \tag{5}$$

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* *Ueber Tchebychefsche Annäherungsmethoden*, Dissertation, Göttingen, 1902.

† Cf., e. g., M. Riesz, *Eine trigonometrische Interpolationsformel, ...*, JAHRESBERICHT DER DEUTSCHEN MATHEMATIKER-VEREINIGUNG, vol. 23 (1914), pp. 354–368; pp. 359–360.

3. *Uniqueness of the Solution.* Case I: $k>1$. It can be easily shown that

$$|\tfrac{1}{2}(x+y)|^k<\tfrac{1}{2}(|x|^k+|y|^k) \tag{6}$$

if $k>1$ and $x\neq y$. Assuming the existence of two non-identical solutions $\varphi_1(x)=f(x)-P'_{nk}(x)$, $\varphi_2(x)=f(x)-P''_{nk}(x)$, we get, using in (6) the function $\varphi_3(x)=\frac{1}{2}[\varphi_1(x)+\varphi_2(x)]$,

$$\int_a^b p(x)|\varphi_3(x)|^k dx<\tfrac{1}{2}M_{nk}+\tfrac{1}{2}M_{nk}=M_{nk}, \tag{7}$$

which is impossible.

Case II: $k=1$. Proceeding as above, we get the impossible inequality (7), unless $\varphi_1(x)\varphi_2(x)\geqq 0$ for $a\leqq x\leqq b$. Evidently every function of the type

$$\varphi_3(x)=h\varphi_1(x)+l\varphi_2(x)=\varphi_1(x)+l\psi(x), \tag{8}$$

where $\psi(x)=\varphi_2(x)-\varphi_1(x)$, $h,l>0, h+l=1$, is also a solution. Inasmuch as $\varphi_1(x)$ and $\varphi_2(x)$ can never have opposite signs, all roots of $\varphi_3(x)$ are the roots common to $\varphi_1(x)$ and $\varphi_2(x)$, or, what is the same, to $\varphi_1(x)$ and $\psi(x)$. Since $\psi(x)$ is a polynomial of degree $\leqq n$, we conclude:

*If there exist two solutions $\varphi_1(x)$, $\varphi_2(x)$, there exists necessarily a third one $\varphi_3(x)$, which has not more than n roots in (a,b). We shall prove that the last conclusion leads to a contradiction.**

Let the zeros of $\varphi_3(x)$ in (a,b) be

$$x_1<x_2<\cdots<x_m \qquad (m\leqq n). \tag{9}$$

We have necessarily $m>0$; otherwise the function $\varphi(x)=\varphi_3(x)+\eta$ with a properly chosen η gives $I_{nk}<M_{nk}$. Consider now two groups of the roots (9):

$$z_1, z_2, \cdots, z_{m'}; \quad y_1, y_2, \cdots, y_{m''} \quad (m'+m''=m),$$

where the z's are the roots, if any, at which $\varphi_3(x)$ changes sign, and the y's are those at which it does not. Form the polynomial

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* In the case that $f(x)\equiv x^{n+1}$, it was established in my Thesis that the number of roots is $n+1$.

$$(10)\qquad \prod(x) = \pm\prod_{i=1}^{m'}(x-z_i) \quad \left(\prod(x)=1 \text{ for } m'=0\right),$$

where the sign $\pm$ is chosen so that $\prod(x)\varphi_3(x)\geqq 0$ in (a, b). Draw the curve $y = |\prod(x)|$ and the line $y=\varepsilon$ with $\varepsilon>0$ sufficiently small. Project the points of intersection on the x-axis. Thus we get a set of points Z, made up of m' intervals

$$(11)\qquad (z_i-\delta_i', z_i+\delta_i'') \quad (i=1, 2, \cdots, m'),$$

and a complementary set $K=b-a-Z$, such that

$$(12)\qquad \varepsilon = \max\left|\prod(x)\right| \text{ on } Z = \min\left|\prod(x)\right| \text{ on } K.$$

It is evident that for ε sufficiently small all maxima points of the curve and all y's are outside Z, and

$$(13)\qquad \int_K p(x)\,dx > 3\int_Z p(x)\,dx.$$

The numbers ε and δ_i', δ_i'' being fixed, we take $\delta>0$ so small that the set of points

$$(14)\qquad Y:\ y_j-\delta\leqq x\leqq y_j+\delta \quad (j=1, 2, \cdots, m'')$$

is outside Z, and

$$(15)\qquad \begin{aligned} &\int_Y p(x)\,dx < \frac{\varepsilon}{2M}\left(\int_K p(x)\,dx-\int_Z p(x)\,dx\right),\\ &\int_Y p(x)\,dx < \int_Z p(x)\,dx, \end{aligned}$$

where M is the maximum of $|\prod(x)|$ in (a, b). Since $\varphi_3(x)$ is continuous and has no zeros in

$$(16)\qquad K' = b-a-Z-Y = K-Y,$$

we have

$$(17)\qquad |\varphi_3(x)|\geqq h'>0 \quad \text{on } K'.$$

We form now the function

$$(18)\qquad \varphi(x) = \varphi_3(x)-\eta\prod(x), \qquad 0<\eta<h'/M,$$

and observe that

$$
(19) \qquad \begin{cases} |\varphi(x)| = |\varphi_3(x)| - \eta \left|\prod(x)\right| & \text{on } K', \\ |\varphi(x)| \leqq |\varphi_3(x)| + \eta \left|\prod(x)\right| & \text{on } Z+Y, \end{cases}
$$

whence, by combination with (12), (13), and (15), we get the desired contradiction:

$$
\int_a^b p(x)|\varphi(x)|\,dx < \int_a^b p(x)|\varphi_3(x)|\,dx = M_{n1}.
$$

The existence proof gives incidentally the following corollary.

COROLLARY. *To an arbitrarily large $A > 0$ there correspond certain K_i $(i = 0, 1, 2, \cdots, n)$ such that any one of the inequalities $|u_i| > K_i$ implies that*

$$
\int_a^b p(x)\left|f(x) - \sum_{i=0}^{n} u_i x^i\right|^k dx > A,
$$

where $P(x)$ and $f(x)$ have the properties specified, and $k \geqq 1$.

4. *Tchebychef Approximating Polynomials.* THEOREM II. *Let $f(x)$ be continuous in (a, b), $p(x) \geqq 0$, and*

$$
\int_\alpha^\beta p(x)\,dx > 0
$$

whenever $a \leqq \alpha < \beta \leqq b$. Then, for $k \to \infty$, the minimizing polynomial $P_{nk}(x)$ tends uniformly to the polynomial $T_n(f)$, of degree $\leqq n$, which gives the closest approximation to $f(x)$ in the interval (a, b), in the sense of Tchebychef.

The proof is very similar to that given by G. Pólya, loc. cit. We assume, without loss of generality,

$$
(20) \qquad \int_a^b p(x)\,dx = 1,
$$

replacing, if necessary, $p(x)$ by $cp(x)$, $c > 0$ being properly chosen. We have evidently, taking $k > 1$,

$$
(21) \qquad \begin{aligned} M_{nk} &= \min I_{nk} < E_n^k, \\ E_n &= \max|f(x) - T_n(f)| \text{ for } a \leqq x \leqq b. \end{aligned}
$$

If $a < \alpha < \beta < b$, and $|f(x) - P_{nk}(x)|$ attains its minimum in (α, β) at $x = z$, then we get, using (21),

$$
(22)\qquad
\begin{gathered}
|P_{nk}(z)| < \frac{E_n}{\sqrt[k]{\int_a^b p(x)\,dx}} + \Phi < \frac{E_n}{\int_a^b p(x)\,dx} + \Phi, \\
\Phi = \max |f(x)| \text{ in } (a, b).
\end{gathered}
$$

We form now $n+1$ partial intervals (α_i, β_i) all belonging to (a, b) and separated by segments of a certain length δ, and we use (22). *Since the right-hand member in* (22) *does not depend on* k, we can follow the reasoning of G. Pólya and prove (a) the coefficients in $P_{nk}(x)$, considered as functions of k, are bounded, so that the sequence (P_{nk}) admits one or several limiting polynomials; (b) all these limiting polynomials coincide with the polynomial of best approximation $T_n(f)$, which is known to be unique.

COROLLARY. *The polynomial minimizing the integral*

$$
\int_a^b p(x)\,|x^n + \cdots|^k dx
$$

tends for $k \to \infty$ *uniformly to*

$$
\frac{1}{B}\cos(n \text{ arc} \cos y), \qquad y = \frac{2x - a - b}{b - a},
$$

for every $p(x)$ *satisfying the conditions of Theorem* II, *where* $B(= 2^{2n-1}/(b-a)^n)$ *is the coefficient of* x^n *in* $\cos(n \text{ arc} \cos y)$.

In fact, $x^n - (1/B)\cos(n \text{ arc} \cos y)$ is the polynomial of degree $\leq n-1$ giving the best approximation to x^n in (a, b).

The results given above hold if we replace polynomials by trigonometric sums, provided $p(x)$ and $f(x)$ are periodic functions.

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