CRITERIA THAT ANY NUMBER OF REAL POINTS IN n-SPACE SHALL LIE IN AN $(n-k)$-SPACE

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The object of the present paper is to establish an algebraic identity from which may be deduced necessary and sufficient conditions that any large number of real points in n-dimensional linear space shall lie in a linear $(n-k)$-space.

Let the following matrix, in which the number of columns is m and the number of rows is $n+1\,[m \geqq (n+1)]$, be compounded with its conjugate:

$$\begin{matrix} 1 & 1 & \cdots & 1 \\ x_{1,1} & x_{2,1} & \cdots & x_{m,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{m,2} \\ \cdot & \cdot & \cdots & \cdot \\ x_{1,n} & x_{2,n} & \cdots & x_{m,n}. \end{matrix}$$

The determinant of the resulting symmetric square array is

$$\begin{vmatrix} m & \Sigma x_{i,1} & \Sigma x_{i,2} & \cdots & \Sigma x_{i,n} \\ \Sigma x_{i,1} & \Sigma x_{i,1}x_{i,1} & \Sigma x_{i,1}x_{i,2} & \cdots & \Sigma x_{i,1}x_{i,n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma x_{i,n} & \Sigma x_{i,n}x_{i,1} & \Sigma x_{i,n}x_{i,2} & \cdots & \Sigma x_{i,n}x_{i,n} \end{vmatrix} \equiv \Delta;$$

$$(i = 1, 2, 3, \cdots, m).$$

Multiply all of the rows of Δ except the top row by m, compensate by prefixing m^{-n}, and remove the factor m now common to the constituents of the first column to get

$$\Delta = m^{1-n}\begin{vmatrix} 1 & \Sigma x_{i,1} & \Sigma x_{i,2} & \cdots & \Sigma x_{i,n} \\ \Sigma x_{i,1} & m\Sigma x_{i,1}x_{i,1} & m\Sigma x_{i,1}x_{i,2} & \cdots & m\Sigma x_{i,1}x_{i,n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \Sigma x_{i,n} & m\Sigma x_{i,n}x_{i,1} & m\Sigma x_{i,n}x_{i,2} & \cdots & m\Sigma x_{i,n}x_{i,n} \end{vmatrix};$$

$$(i = 1, 2, 3, \cdots, m).$$

Next subtract $\sum_{i=1}^{i=m} x_{i,k}$ times the first column from the $(k+1)$th column, $(k=1, 2; 3, \cdots, n)$, in order to reduce to zero all the constituents of the top row, except the leading constituent, and to find $\Delta = U_n/m^{n-1}$, where

$$U_n \equiv \begin{vmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdot & \cdot & \cdot & \sigma_{1,n} \\ \sigma_{2,1} & \sigma_{2,2} & \cdot & \cdot & \cdot & \sigma_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \sigma_{n,1} & \sigma_{n,2} & \cdot & \cdot & \cdot & \sigma_{n,n} \end{vmatrix}$$

and

$$\sigma_{p,q} \equiv m \sum_{i=1}^{i=m} x_{i,p} x_{i,q} - \left(\sum_{i=1}^{i=m} x_{i,p}\right)\left(\sum_{i=1}^{i=m} x_{i,q}\right)$$
$$= \sum_{i=j+1}^{i=m} \sum_{j=1}^{j=m-1} [(x_{i,p}-x_{j,p})(x_{i,q}-x_{j,q})] = \sigma_{q,p}.$$

Now the determinant Δ produced by compounding the matrices specified above is known to equal the sum of the squares of all the ν determinants of order $n+1$ that can be formed from the columns of the original matrix, where

$$\nu \equiv \binom{m}{n+1}.$$

Let any one of these determinants be denoted by D_r; then the required algebraic identity is

$$U_n = m^{n-1} \sum_{r=1}^{r=\nu} (D_r^2). \tag{1}$$

Thus far no special meaning has been assigned to the x's; they may represent complex quantities, etc.

To obtain the criteria contemplated advantage will be taken of the fact that D_r is squared in identity (1) so that if the x's are real numbers D_r^2 will be incapable of becoming negative. Accordingly let the rectangular coordinates of a system of real points in n-dimensional flat space be

$$(x_{i,1}, x_{i,2}, \cdots, x_{i,n});\ i = 1, 2, 3, \cdots, m;\ m \geqq (n+1).$$

Also let S_t symbolize a linear space of t dimensions, a t-flat.

Now the vanishing of $\sum(D_r^2)$ is a necessary and sufficient condition that the m given real points shall lie in the same S_{n-1}, hence, by formula (1), *a necessary and sufficient condition that any number* $m\,[\geqq(n+1)]$ *of real points in* S_n *shall lie in the same* S_{n-1} *is the vanishing of* U_n.

When $m=n+1$, $\nu=1$ so that there is only one D_r in $\sum(D_r^2)$. This D_r represents $n!$ times the content of the hyper-figure or simplex having the $n+1$ given points as vertices.* Hence, for $m>(n+1)$, $\sum(D_r^2)$ is proportional to the sum of the squares of the contents of all the simplexes that can be formed from the m points taken $n+1$ at a time as vertices of each geometric figure. Accordingly the above italicized statement may also be interpreted as meaning that the contents of all the simplexes involved vanish.

Keeping $m=n+1$, and giving n successively the values $1, 2, 3, 4, \cdots, n$, we may derive from the identity (1) the following expressions for the respective magnitudes of the length of a segment in S_1, the area of a triangle in S_2, the volume of a tetrahedron in S_3, the hyper-volume of a pentahedroid in $S_4, \cdots$, the content of a simplex in S_n:

$$\sigma_{1,1}^{1/2}, \quad \frac{|\sigma_{1,1},\ \sigma_{2,2}|^{1/2}}{2\sqrt{3}}, \quad \frac{|\sigma_{1,1},\ \sigma_{2,2},\ \sigma_{3,3}|^{1/2}}{24},$$

$$\frac{|\sigma_{1,1},\ \sigma_{2,2},\ \sigma_{3,3},\ \sigma_{4,4}|^{1/2}}{120\sqrt{5}}, \cdots, \frac{|\sigma_{1,1},\ \sigma_{2,2},\ \cdots,\ \sigma_{n,n}|^{1/2}}{n!\,(n+1)^{(n-1)/2}}.$$

The extension of the above italicized statement from S_{n-1} to S_{n-k} is an immediate consequence of the well known properties of orthogonal projections of linear spaces. The fundamental idea is that identity (1) holds for a smaller number of coordinates than n and hence it may be applied to the orthogonal projections of the m given points upon *all* of the

$$\binom{n}{n-k+1}$$

* P. H. Schoute, *Mehrdimensionale Geometrie*, Part 2, §§ 36, 37.

coordinate-S_{n-k+1}'s. In other words the original matrix is to be replaced by

$$\binom{n}{n-k+1}$$

matrices having the same top row of m 1's while the remaining rows are composed of $n-k+1$ of the original rows of x's. There will now be

$$\binom{n}{n-k+1}$$

new systems of points,—one in each coördinate-S_{n-k+1},—to all of which the above italicized test must be applied. The orders of the U_n's and D_r's of formula (1) will be $n-k+1$ and $n-k+2$ respectively. Without further comment it should be perfectly clear that *necessary and sufficient conditions that any number of real points in n-dimensional flat space shall lie in an $(n-k)$-dimensional flat space are that all the*

$$\binom{n}{n-k+1}$$

determinants U of order $n-k+1$ in the σ's shall vanish while one, at least, of the determinants U of order $n-k$ shall be finite.

The last sentence may be stated in terms of the rank of the U of order n.* Incidentally the writer has found it possible to express the general criteria analytically in terms of only two determinants involving polynomial constituents composed of the σ's.

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* G. Kowalewski, *Determinantentheorie*, § 52.