GROUPS IN WHICH THE NORMALISER OF EVERY ELEMENT EXCEPT IDENTITY IS ABELIAN*

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1. *Introduction.* Certain groups, of which the tetrahedral and icosahedral groups are non-trivial examples, possess the property that the normaliser of every element except identity is abelian. In this paper we shall investigate the properties of such groups.

2. *Abelian Subgroups.* An abelian subgroup of a group will be called a *maximal* abelian subgroup if it is not contained in any larger abelian subgroup of the group.

If two abelian subgroups of a group G, in which the normaliser of every element except identity is abelian, have an element besides identity in common, they generate an abelian group; for an element common to the abelian subgroups H and I is invariant under (H,I), which must therefore be abelian. It follows that *the maximal abelian subgroups of G are independent*; that is, no two maximal abelian subgroups of G have an element in common besides identity.

Since every prime-power group possesses an invariant element besides identity, *the Sylow subgroups of G are abelian.* Moreover, *the Sylow subgroups of G are independent.* For if two Sylow subgroups H and I (necessarily of the same order p^a) had an element in common besides identity, (H,I) would be an abelian subgroup of G of order $p^b > p^a$, which contradicts the fact that p^a is the highest power of p that divides the order of G.

Let H be a maximal abelian subgroup of G, I a Sylow subgroup of H, and J that Sylow subgroup of G which includes I. Since H and J have I in common, (H,J) is

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an abelian group. But H is a *maximal* abelian subgroup of G. Hence H contains J. Therefore *the Sylow subgroups of a maximal abelian subgroup of G are Sylow subgroups of G.* It follows that *the orders of two non-conjugate maximal abelian subgroups of G are relatively prime.* If G possesses exactly r conjugate sets of maximal abelian subgroups, of which $H_1, H_2, \cdots, H_r$, of orders $h_1, h_2, \cdots, h_r$ respectively, are representatives, then every element of G except identity is included in one and only one conjugate of one of the H's, and the order of G is $h_1 h_2 \cdots h_r$, $h_1, h_2, \cdots, h_r$ being relatively prime.

If a maximal abelian subgroup H, of order h, has y conjugates under G, the total number of elements of G whose orders divide h is $y(h-1)+1$; for every element whose order divides h is included in a conjugate of H, and the conjugates of H are independent. By Frobenius' theorem,

$$y(h-1)+1 \equiv 0, \pmod{h},$$

whence

$$y \equiv 1, \pmod{h}.$$

Therefore, *the number of conjugates of a maximal abelian subgroup of order h is of the form $1+xh$.*

3. *Solvable Groups.* Let G be a solvable group in which the normaliser of every element except identity is abelian. A minimal invariant subgroup I is an abelian group. The maximal abelian subgroup H which contains I must be invariant under G; otherwise I would be included in H and a conjugate of H, whereas the maximal abelian subgroups of G are independent.

Arrange G in cosets as regards H. Every element of G transforms H into itself, thus establishing an isomorphism of H. Elements in the same coset establish the same isomorphism, while elements in different cosets establish different isomorphisms. Since no element of H except identity is commutative with an element of G not in H, G/H is simply isomorphic with a group of isomorphisms of H, each of which leaves no element of H unchanged except identity. It follows* that G/H is either a cyclic group or a dicyclic group. If G/H is dicyclic, it contains a non-abelian Sylow subgroup of even order and G contains a subgroup simply isomorphic with this Sylow subgroup, since the order of H is prime to the order of G/H. But the Sylow subgroups of G are abelian. Hence G/H is a cyclic group.

Thus *a solvable group in which the normaliser of every element except identity is abelian contains an invariant maximal abelian subgroup, and the corresponding quotient group is cyclic.*

4. *Insolvable Groups.* An insolvable group G in which the normaliser of every element except identity is abelian must contain a minimal invariant subgroup H, which is a simple group of composite order. If G is larger than H, let I be a subgroup of G which contains H, such that H is of prime index p in I.

Suppose that no element of H is commutative with an element of I which is not in H. A maximal abelian subgroup J of H must also be a maximal abelian subgroup of I; otherwise, H would contain two non-conjugate abelian subgroups of the same order, one of which is a maximal abelian subgroup, which conflicts with a theorem proved in § 2. Therefore the normaliser K of J in I is larger than, and includes, the normaliser L of J in H. By § 3, K/J is a cyclic group. Since J is a maximal abelian subgroup of K, the order of J is prime to the order of K/J. Hence K contains an element t whose order n equals the order of K/J, and $K=(t, J)$. Since L is a subgroup of K which includes J, $L=(t^a, J)$, where a is a divisor of n. Since t^a is an element of H which is commutative with t, $t^a=1$, and $L=J$. It follows that J is the normaliser in H of every Sylow subgroup of J. Since the Sylow subgroups of J are Sylow subgroups of H, H is not a simple group,† contrary to supposition.

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* Burnside, *Theory of Groups*, 2d edition, p. 336, Theorem V.

† Burnside, p. 327, Theorem II.

It follows that I contains an element t not in H, which is commutative with some element of H other than identity. Let J be the normaliser of t in I, and K the subgroup common to J and H. Since the maximal abelian subgroups of H and of I are independent, while J and K are maximal abelian subgroups of I and H respectively, the number of conjugates of J under I equals the number of conjugates of K under H. Let this number be x, and let h, i, j, k denote the orders of H, I, J, K respectively. The elements of H whose orders divide $j = kp$ are included in the conjugates of K, while the elements of I whose orders divide kp are included in the conjugates of J. Observing that the order of every element of I not in H divides kp, we have

$$x\,(kp-1)+1 = x\,(k-1)+h\,(p-1),$$

from which $x = h/k$. Therefore K is the normaliser of K in H. It follows as in the preceding paragraph that H is not a simple group.

Since the assumption that G is larger than H leads to a contradiction, it follows that *an insolvable group in which the normaliser of every element except identity is abelian is a simple group.*

The linear fractional group of order $2^n\,(2^{2n}-1)$, $(n>2)$ is an example of a simple group in which the normaliser of every element except identity is abelian. That this group actually possesses the property in question follows from its analysis.*

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* Dickson, *Linear Groups,* pp. 262–265.