THE FEBRUARY MEETING IN NEW YORK

The two hundred fortieth regular meeting of the Society was held at Columbia University, on Saturday, February 28, 1925, extending through the usual morning and afternoon sessions. The attendance included the following sixty-one members of the Society:

Alexander, R. L. Anderson, C. R. Ballantine, J. P. Ballantine, Birkhoff, Blake, Bowden, R. W. Burgess, G. A. Campbell, Alonzo Church, Cole, Dadourian, Eisenhart, Eshleman, Fite, Frink, Fry, Gafafer, Gehman, Gronwall, C. C. Grove, Guggenbühl, Hausle, Hille, Himwich, Louis Ingold, Joffe, Kasner, O. D. Kellogg, Kline, Lamson, Langer, Langman, Lefschetz, Harry Levy, McGiffert, McMackin, MacColl, MacNeish, Meder, Mirick, Molina, C. L. E. Moore, Mullins, Olson, Pell, Pfeiffer, R. G. Putnam, Rainich, Reddick, Ritt, Schelkunoff, Seely, Siceloff, J. H. Taylor, H. D. Thompson, Veblen, H. E. Webb, Wedderburn, Whittemore, Wiener.

There was no meeting of the Council or of the Trustees.

At the beginning of the afternoon session, a paper was read by Professor J. W. Alexander, at the request of the Program Committee, on *Problems in the topological theory of manifolds.*

Professor Béla de Kerékjártó, of the University of Szeged, Hungary, spoke briefly of the mathematical journal recently founded at that university, the ACTA LITTERARUM AC SCIENTIARUM REGIAE UNIVERSITATIS HUNGARICAE FRANCISCO-JOSEPHINAE, SECTIO MATHEMATICARUM; the articles in this journal are in French or German. He suggested that members of the Society might call the attention of their university libraries to this publication.

President Birkhoff presided at both sessions.

Titles and abstracts of the papers read at this meeting follow below. Professor Alexander's second paper and the papers of Bennett, Dodd, Franklin, Garabedian, James, Jeffery, Libman, Osgood, Reynolds, Thomas, and Weisner were read by title; Professor Morley's paper was communicated by Mr. Rainich. Professor de Kerékjártó was introduced by President Birkhoff.

1. Professor C. L. E. Moore: *Minimal varieties of two or three dimensions whose element of arc is a perfect square.*

After determining the euclidean space of lowest dimension in which minimal varieties of two or three dimensions whose element of arc is a perfect square can lie, the author discusses the intrinsic geometry of these varieties.

2. Professor L. P. Eisenhart: *Fields of parallel vectors in a Riemannian geometry.*

In accordance with the definition introduced by Levi-Civita, parallelism of vectors depends upon the metric of the space and upon the curve along which a vector is displaced so as to remain parallel to its initial position. In a general Riemannian space there does not exist a field of vectors which are parallel to one another regardless of the curves of displacement. However, there are certain Riemannian spaces in which there are one, or more, fields of vectors possessing this property. In this paper canonical forms of the metric of these spaces are determined.

3. Mr. G. Y. Rainich: *On the Riemann tensor.*

In a flat space, to every contour corresponds its *tensor area*, an alternating tensor of rank two. If a curved space $\sum$ is immersed in our flat space, we consider for every contour on $\sum$ its *contour variation of normals* (normal flat spaces). For infinitesimal contours in the vicinity of a given point, the contour variation of normals is a linear function of the tensor area, and this function is the *Riemann tensor* at the point considered. By integrating (in a sense previously introduced) the Riemann tensor over a surface, we obtain again the finite variation of normals corresponding to the contour limiting this surface. If the curved space is four-dimensional, the Riemann tensor is the sum of two parts, of which the first assigns to absolutely perpendicular planes equal curvatures and the second opposite curvatures. The first part involves 11 and the second 9 constants. The second part has the same coefficients as the "cosmological energy tensor"; its vanishing is equivalent to the cosmological equations. In physical space-time, the second part depends only on 5 constants and can be expressed quadratically through the electromagnetic tensor.

4. Mr. G. Y. Rainich: *Integrals in curved space.*

It has been previously shown how integrals can be formed when a tensor field has superfluous indices; this required the consideration of the containing flat space. However, when the tensor field depends on the superfluous indices in the same way as on the others, it is possible to form integral expressions without leaving the curved space, by presenting the tensor as the sum of products of tensors of which integrals can be taken and by forming the sum of the products of these integrals. This is first applied to the case of the indicator of a surface; it is shown that only for a minimal surface do the expressions considered vanish over a reducible contour. For an irreducible contour on a one-sided surface they give rise to residues; the residues are the same for all simple irreducible contours of a given surface. In the case of the physical space-time, the second part of the Riemann tensor when treated in a similar manner gives an expression which vanishes on a reducible closed surface, and for a surface surrounding a singularity it gives the square of the electric charge.

5. Professor Frank Morley: *Comitants of a curve under inversion.*

This paper deals with the comitants under inversion of a plane curve. The equation of such a curve, expressed in circular coordinates, is obtained by bordering a self-conjugate or hermitian matrix. The determinant of this matrix is an invariant. If with the curve we combine the special curve which represents a repeated point, the determinant of the pencil gives the reciprocal curve of the given one. The pencil formed by a curve and its reciprocal gives a series of invariants. The results are included in the memoir of Kasner (TRANSACTIONS OF THIS SOCIETY, vol. 1) but it may be of interest to consider this important question from all points of view.

6. Professor Edward Kasner: *Null geometry.*

A null manifold (all distances vanishing) of k dimensions cannot exist in a euclidean space of less than $2k$ dimensions. In space of $2k$ dimensions, null k-flats exist, and in higher spaces curved null spaces exist.

7. Professor Louis Ingold: *Extensions of the equations of Gauss and Codazzi.*

The paper considers Riemann spaces R_n determined by a general fundamental vector $F(u^1, u^2, \cdots, u^n)$. The first partial derivatives $\partial f/\partial u^i = f_i$ are regarded as independent tangent vectors to R_n. The first fundamental quantities E_{ij} are the scalar products $f_i f_j$. Second fundamental quantities L_{ijrs} are defined as the scalar products $N_{ij}\, N_{rs}$ of vectors normal to R_n, where N_{ij} is linear in the corresponding f_{ij} and the tangent vectors f_k. Third fundamental quantities are defined in a similar way from normals N_{ijk} linear in f_{ijk} and the normals and tangents defined above. These are orthogonal to the N_{rs}. In case the vector f satisfies a completely integrable system of $n-1$ linear differential equations of the second order, all of the normals N_{ij} are expressible in terms of a single normal N; and the first and second fundamental quantities satisfy the usual equations of Gauss and Codazzi. If it is assumed that this is not true, but that the normals N_{ijk} are expressible in terms of a single normal vector, then analogous necessary relations connecting the first, second, and third fundamental quantities are obtained. Further extensions are obvious.

8. Dr. Harry Levy: *Tensors determined by a hypersurface in a Riemann space.*

Consider a hypersurface, V_n, of n dimensions, embedded in a general Riemann space of $n+1$ dimensions. If we let V_n be represented by the equation $u^0 = 0$ and if we take for the parameter u^0 the arc of the geodesics normal to V_n, the linear element of the space becomes $ds^2 = (du^0)^2 + C_{\alpha\beta}du^\alpha du^\beta$ $(\alpha, \beta = 1, 2, \cdots, n)$. We find that the coefficients in the expansion of $C_{\alpha\beta}$ in a power series in u^0 are a sequence of tensors intimately connected with the geometry of the hypersurface. In the first part of this paper we deduce a number of identities which hold in general tensor analysis. In the second part we consider these tensors and obtain general properties of Riemann space.

9. Dr. Harry Levy: *Symmetric tensors of the second order whose covariant derivatives vanish.*

We give a complete geometric characterization of tensors of the kind mentioned in the title, showing that necessary and sufficient conditions that the covariant derivative of b_{ij} vanish are that b_{ij} be a first integral of the differential equations of the geodesics of the space, and that the space whose fundamental form is $b_{ij}\,dx^i\,dx^j$ admit of geodesic representation upon the given space. We prove further that, in a space of constant curvature, a tensor whose covariant derivatives are zero is a constant multiple of the fundamental tensor.

10. Professor H. L. Olson: *Congruences with constant absolute invariants.*

This paper is based on the researches of Wilczynski, chiefly on his memoir *Sur la théorie générale des congruences,* published by the Royal Academy of Belgium. Regarding the special type of congruences considered, it is shown that both sheets of the focal surface have constant absolute invariants, and that the cuspidal edges of the developables are two families of projectively equivalent anharmonic curves. The Laplace transforms of any such congruence have constant absolute invariants. Necessary and sufficient conditions that a congruence be projective to its 1st or (—1)st Laplace transform are that its absolute invariants be constants and that $\overset{(z)}{\mathfrak{B}} - \overset{(y)}{\mathfrak{B}}$ and $\overset{(y)}{\mathfrak{C}}'' - \overset{(z)}{\mathfrak{C}}''$, defined by Wilczynski, vanish identically. Any congruence of the latter type is contained in a tetrahedral complex or one of the limiting forms of the tetrahedral complex.

11. Professor W. F. Osgood: *On normal forms of differential equations.*

Linear differential equations of the second order (and their Schwarzian resolvents) are considered on an algebraic configuration which is first taken in the form of Noether's normal curve. The latter is now projected on a pencil of hyperplanes, and the algebraic configuration is treated (i) in the binary domain thus arising; and (ii) in the domain corresponding to a uniformization by means of automorphic functions with limiting circle. The invariant, to the existence of which Klein was led in the case of the group of the binary linear transformations of Case (i), is obtained explicitly and independently in terms of the uniformizing variables of Case (ii), and its properties are read off by means of the automorphic functions.

12. Professor A. A. Bennett: *Two new arctangent relations for* π.

In this paper the author gives two new formulas for use with Gregory's series in computing π. The first is not unsuited for pencil and paper computation, and gives much more rapidly convergent series than does Machin's well known relation. The second is intended for use with a computing machine. The first may be written in the form $\pi/4 = 12 \operatorname{arcctn} 18 + 3 \operatorname{arcctn} 70 + 5 \operatorname{arcctn} 99 + 8 \operatorname{arcctn} 307$. The second has more arguments, but has 200 for its smallest argument.

13. Professor A. A. Bennett: *Diophantine arccotangent relations.*

In this paper the author shows that a necessary and sufficient condition that the equation $\operatorname{arcctn} x_1 + \operatorname{arcctn} x_2 = \operatorname{arcctn} y_1 + \operatorname{arcctn} y_2$ be satisfied by given integers is that certain corresponding expressions u_1, u_2, v_1, v_2 satisfy the congruences $u_1 v_1 \equiv u_1 v_2 \equiv u_2 v_1 \equiv u_2 v_2 \pmod{x_1 + x_2}$. Several infinite systems of solutions are obtained, and a complete table of all solutions, up to $x_1 + x_2 = 25$, is given. The simpler case of the three term arccotangent relation is disposed of. The problem in both cases reduces to a utilization of a table of numerical factors of $x^2 + 1$, and the theory of the construction and checking of such a table is developed.

14. Professor E. L. Dodd: *The fitting of curves by the use of moments and conjugate moments.*

If f is the frequency of x, the conjugate rth moment is defined as $\sum f \cdot x^r \cdot \operatorname{sign} x$, with $r = 0, 1, 2, \cdots$. The author shows that conjugate moments are especially useful in curve-fitting, since double the usual number of parameters may be introduced. Suppose, for example, that the frequency function is $y = e^{-x^2/2}\,(A + Bx + Cx^2 + \cdots + Kx^{2s+1})$, where the arithmetic mean has been made the origin and the standard deviation the unit. Using moments up to the sth with their conjugates, the $s+1$ alternate parameters $A, C, E, \cdots$ may be obtained by solving $s+1$ linear equations, and the remaining parameters likewise. Thus we may fit a six-parameter curve to a distribution without going higher than second moments (it seems especially advisable to avoid high moments when extreme variates are irregular) while even twenty-two-parameter curves become tractable with the Pearson tables.* For comparison, the distribution of 1130 pensioners exhibited by Arne Fisher† was fitted by a six-parameter curve of the above form (with corrections for grouping) and 13 positive and 13 negative aberrations were obtained, the same absolute total (26) as was obtained by Fisher by the Charlier method involving moments up to the fourth.

15. Dr. E. E. Libman: *Linear complex of conics.*

A conic in space involves in its specification eight arbitrary constants. When these are expressed as integral linear non-homogeneous functions of three independent parameters, the system so obtained is called a linear complex. The linear complex of conics so defined has the following properties: (I) There is one and only one conic upon each plane in space. (II) The centers of the degenerate conics of the complex lie upon a quadric surface and their planes envelope another quadric surface; these are called the singular quadrics of the complex. (III) The conics through a point are in one to one correspondence with the points of a plane quartic curve associated with the point. The plane of the quartic, and the point are called polar and pole with respect to the complex. (IV) The points that lie upon their polars are on one of the singular quadrics, while the planes that contain their poles envelope the other.

The machinery used in the investigation is quaternions with the usual Hamiltonian symbols.

16. Professor C. N. Reynolds: *On the map-coloring problem, with particular reference to connected sets of pentagons.*

In this paper the author considers the problem of developing, by analytic methods, the implications of such geometric reductions of the map-coloring problem as have been published; i. e., those due to Kempe, Birkhoff, and Franklin, but not those of Errera. His method is that of applying linear difference equations to the study of the topological properties of irreducible connected configurations of pentagons. It is then shown that the reductions considered imply the possibility of coloring, in four colors, the map

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* *Tables for Statisticians and Biometricians*, Tables II and IX.

† *Frequency Curves*, 1922, p. 48.

of any spherical surface which is divided into not more than twenty-seven regions. Two maps of twenty-eight regions each which can be colored in four colors but which are irreducible, in the sense defined above, are then exhibited as a check on the closeness of fit between the analytic methods presented and the geometric problem considered.

17. Professor C. A. Garabedian: *Solution of the problem of the thick rectangular plate, clamped or supported at its edges and under uniform or central load.*

The author has already given a solution of the problem of the thick rectangular plate supported at its edges and under uniform pressure.* The method employed may be successfully applied to all four problems mentioned in the title above, and it is noteworthy that the resulting formulas of displacement are in each case simply expressible in terms of the solution (and its derivatives) of the corresponding thin plate problem. Since the four thin plate solutions at issue are given in complete form in Hencky's thesis,† rigorous and usable formulas for the thick plate are now available to the engineer. The solutions of the present note are analogous to the author's solutions of the same problems in circular plates,‡ and there are certain interesting remarks applicable to both rectangular and circular cases. Finally the method is capable of including plates of variable thickness. An abstract of results will be found in the COMPTES RENDUS of January 26, 1925; details will be given in an extended paper now in preparation.

18. Professor Glenn James: *A complete solution of the cubic equation.*

This paper determines an interval in which one and only one of the roots of a cubic equation lies, and then developes an interpolation theorem concerning a root in a given interval. Upon the basis of this theorem, a root is secured as a function of the coefficients which can be evaluated for numerical equations by a certain recurrence formula. The other roots are obtained in general form from the reduced equation.

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* COMPTES RENDUS, vol. 178 (1924), p. 261.

† *Der Spannungszustand in rechteckigen Platten,* Darmstadt, 1913 (published by Oldenbourg, Munich).

‡ TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), pp. 379–388.

19. Professor R. L. Jeffery: *Functions of two variables for which the double integral does not exist.*

It is pointed out by E. W. Hobson* that even though a function of two variables is continuous in each variable separately, the double integral in the sense of Riemann need not exist. This is obviously true if the function is unbounded, but there seems to be nowhere in the literature a bounded function which illustrates Hobson's observation. In this note, a function $f(x, y)$ is constructed on the unit square which is bounded and continuous in each variable separately, but which is discontinuous in the two variables with a saltus equal to unity at every point of a two-dimensional set whose plane content is $\frac{1}{4}$. Hence, for this function, the Riemann double integral fails to exist.

20. Dr. Louis Weisner: *On the number of elements in a group which have a power in a given conjugate set.*

This paper will appear in full in an early number of this BULLETIN.

21. Dr. J. M. Thomas: *The number of even and odd absolute permutations of n letters.*

This paper appears in full in the present number of this BULLETIN.

22. Dr. J. M. Thomas: *Note on the projective geometry of paths.*

The projective curvature tensor discovered by Weyl (GÖTTINGER NACHRICHTEN, 1921, p. 99) is derived in a manner analogous to the derivation of the ordinary curvature tensor, and a new projective tensor of rank 13 is obtained.

23. Professor J. W. Alexander: *Problems in the topological theory of manifolds.*

This paper will appear in an early issue of this BULLETIN.

24. Professor J.W. Alexander: *Combinatorial analysis situs.*

The author determines a necessary and sufficient condition for the homeomorphism of two complexes which is expressed in terms of cells and their incidence relations rather than

* *Theory of Functions of a Real Variable,* 2d edition, vol. 1, § 365.

in terms of points and limit points. He is then able to develop the theory of the connectivity numbers and intersection invariants without further appeal to continuity considerations.

25. Professor O. D. Kellogg: *Note on the convergence of real power series representing harmonic functions.*

If a series of spherical harmonics in x, y, and z converges within the sphere $x^2+y^2+z^2<R^2$, then the power series in x, y, and z which represents the same function converges within the octahedron $|x|+|y|+|z|<R$. A corresponding result holds in the case of n variables. For $n=2$, the complete two-dimensional region of convergence is thus characterized, as has been shown by Bôcher. For $n>2$, however, the complete n-dimensional region is not, in general, given by the theorem indicated above.

26. Professor J. F. Ritt: *Transcendental transcendency of certain functions of Poincaré.*

Poincaré, in 1890, established the existence of a large class of meromorphic functions $y(x)$ for which an $m \neq 1$ exists such that $y(mx)$ is a rational function of $y(x)$. The functions e^x, $\cos x$, and $\wp(x)$ belong to this class. In the present paper, it is proved that with the exception of the three functions just named and certain others closely related to them, the functions of Poincaré are transcendentally transcendental, that is, they do not satisfy algebraic differential equations.

27. Professor J. R. Kline: *Concerning the sum of a countable infinity of mutually exclusive continua.*

Sierpinski has proved that no bounded closed connected set can be the sum of a countable infinity of mutually exclusive closed connected sets. In her dissertation, Miss Anna Mullikin gave an example of a connected set which is the sum of a countable infinity of mutually exclusive closed connected sets. In the present paper it is proved that if S is a point set which is the sum of a countable infinity $S_1, S_2, \cdots$ of closed connected sets having the property that given any ϵ, greater than zero, there are at most a finite number of the sets $S_1, S_2, \cdots$ of diameter greater than ϵ, then S is not connected.

28. Dr. H. M. Gehman: *On extending a continuous* (1,1) *correspondence of two plane continuous curves to a correspondence of their planes.*

It is proved that if any two plane continuous curves are in continuous (1,1) correspondence in such a way that sides of arcs are preserved under the correspondence, then a continuous (1,1) correspondence of their planes can be defined in such a way that the correspondence of the two continuous curves is preserved. This generalizes a theorem previously announced for the case of continuous curves that contain no simple closed curves.

29. Professor S. Lefschetz: *On the problem of inversion of abelian integrals.*

This paper gives a proof of the existence and uniqueness of the solution of Jacobi's inversion problem by means of the implicit function theorem.

30. Dr. Philip Franklin: *Osculating curves and surfaces.*

In this paper several conditions are given under which a curve or surface, obtained as the limit of a sequence of curves or surfaces having a fixed number of points in common with a given curve or surface, osculates this given one. As a typical theorem we have the following: If in some neighborhood of a point P on a curve its defining function possesses a continuous fourth derivative, and if at this point there is a unique osculating conic ($y''' \neq 0$) a sequence of conics having five points in common with the given curve approaches the osculating conic at P when the five points close down on P.

31. Professor R. E. Langer: *On the momental constants of a summable function.*

This paper deals with the momental constants defined by Haskins for any summable function. A set of necessary conditions on the constants of any enumerably infinite set if they are to be the momental constants of a summable function with finite measurable bounds is deduced. These conditions are then shown to be sufficient as well. The determination of the function having a given set of momental constants is made to depend upon the solution of an associated "problem of moments".

32. Professor Norbert Wiener: *The solution of a difference equation by trigonometric integrals.*

In every case in which Nörlund has solved the equation $f(x+1)-f(x) = \varphi(x)$ in real variables, his principal solution may be resolved into the sum of two functions $f_1(x)$ and $f_2(x)$, such that $f_1(x) = -\sum_{s=0}^{\infty}\varphi_1(x+s)$ and $f_2(x) = \int_a^x \varphi_2(z)\,dz + \sum_{\nu=1}^{\infty}(B_\nu/\nu!)\,\varphi_2^{(\nu-1)}(x)$, where $\varphi(x) = \varphi_1(x) + \varphi_2(x)$.

33. Dr. T. H. Gronwall: *On Gibbs' phenomenon.*

This paper investigates the condition for the occurrence of a Gibbs' phenomenon in the Cesàro sums of order k $(0 \leqq k \leqq 1)$ of a Fourier series.

34. Professor Einar Hille: *Some remarks on Dirichlet's series.*

This paper is concerned with certain relations between Dirichlet's, factorial, and binomial series. The starting point is a paper by Wigert (ARKIV FÖR MATEMATIK, ASTRONOMI, OCH FYSIK, vol. 7 (1911), No. 26) in which a binomial series is expressed as the sum of an entire function and a factorial series. The corresponding relation for Dirichlet's series can be obtained from Cahen's expression for such a series by means of a Laplace integral. The desired relation is obtained by applying Pincherle's results on *fonctions déterminates* to this integral. It is observed that Cahen's integral can be used for the purpose of summing a certain class of everywhere divergent Dirichlet's series. Some theorems on the analytic nature of the functions represented by certain Dirichlet's series are given as an application. These results were first derived by Hardy using Borel's method of summation. Various generalizations are considered.

35. Professor Béla de Kerékjártó: *Remarks on convex regions.*

The author discusses the theorems of E. Helly concerning common points of convex regions and of simply connected plane regions.

W. B. FITE,
Acting Secretary.