SCHOUTEN ON RICCI-KALKÜL

Der Ricci-Kalkül. By J. A. Schouten. Berlin, Julius Springer, 1924. 312 pages.

This is Volume X of *Die Grundlehren der Mathematischen Wissenschaften,* edited by R. Courant, and is appropriately dedicated to G. Ricci, founder of the absolute differential calculus, in honor of his seventieth birthday. It is the second book on this subject published by Springer within two years. The authors (Struik and Schouten) are closely associated, and therefore the fact that the two books have many things in common is not surprising. The volume under discussion is more extended than Struik's, the additional space being devoted to a fuller discussion of non-Riemannian geometry. The methods are quite similar, but Schouten does not make use of quite so much symbolism, although Struik attributes most of his symbolism to Schouten.

These volumes give an exceedingly complete account of all the developments in differential geometry of the past five or six years, but the reviewer feels that much of the value, as reference books, is lost on account of the symbolism. The reading of many pages is often required for the full understanding of a single theorem, because it is impossible for one to keep these symbols in mind unless constantly using them.

Many of us feel that one of the best things Ricci did was to invent a consistent notation for covariant and contravariant systems. Ricci is always careful that the upper and lower indices shall have a significance. The ultra modern writers, following Weyl, have departed from this. The best example is $\Gamma_{\lambda\mu}^{\nu}$, which, according to the usual notation, should be a mixed system, covariant in λ, μ and contravariant in ν. Such is not the case, however. I, for one, fail to see why this confusion in notation should have been introduced, as it only makes the reading more laborious. One is compelled to investigate to see just what quantities are covariant or contravariant.

Geometers had been accustomed to think that Riemannian geometry was the most general geometry possible, but in the past few years has come the notion of parallel displacement, which leads to geometries quite different from that of Riemann. In the present volume, Schouten gives a very full account of these new geometries. In fact, only about fifty pages are devoted to Riemannian geometry. It does not take long to convince one that this book is entirely modern both as to material and point of view. Nearly every page has a reference to work which has appeared within six or seven years. A modern mathematical theory usually means one that is not more than fifty or seventy-five years old, and this book, therefore, leaves the pleasant sensation of a really modern theory. The parts of the subject which are not modern are labeled so that one can easily recognise them.

The work is divided into seven chapters. The first gives, from a veıy general point of view, the algebraic part of the ab˞olute calculus. The notions of a vector in a general space and the distinctions between covariant and contravariant vectors is carefully stated. In Chapter II, we begin the analytical part and are introduced at once to the notion of parallel displacement. The displacement is defined by the following properties:

I. The differential of a covariant, contravariant, or mixed quantity, which does not change with change of covariant scale, is a quantity of the same kind.

II. The differential is a linear function of the line elements or

$$\delta\Phi = \Phi_\mu dx^\mu;$$

Φ_μ is a quantity having one more covariant index than Φ and the operation which is represented is symbolized by ∇_μ, hence

$$\Phi_\mu = \nabla_\mu \Phi;$$

$\nabla_\mu \Phi$ is called the covariant derivative of Φ belonging to the displacement.

III. The differential of a sum is the sum of the differentials,

$$\delta(\Phi + \Psi) = \delta\Phi + \delta\Psi,$$

from which follows

$$\nabla_\mu(\Phi + \Psi) = \nabla_\mu \Phi + \Delta_\mu \Psi.$$

IV. The ordinary rules for the differential and therefore of the derivative of a product holds.

V. The differential of a numerical quantity (invariant) is the ordinary differential

$$\delta p = dp, \qquad \nabla_\mu p = \frac{\partial p}{\partial x^\mu}.$$

From these postulates the formulas for differentiation of vectors v^ν, w_λ are easily derived:

$$\nabla_\mu v^\nu = \frac{\partial v^\nu}{\partial x^\mu} + \Gamma^\nu_{\lambda\mu} v^\lambda,$$

$$\nabla_\mu w_\lambda = \frac{\partial w_\lambda}{\partial x^\mu} - \Gamma'^\nu_{\lambda\mu} w_\nu,$$

and similar formulas for quantities of higher order. The character of the displacement then depends on the quantities Γ, Γ', but instead of using them directly the following combinations are used.

Let

$$C_{\mu\lambda}^{\cdot\cdot\nu} = \Gamma_{\lambda\mu}^{\nu} + \Gamma_{\lambda\mu}^{\prime\nu},$$
$$S_{\lambda\mu}^{\cdot\cdot\nu} = \frac{1}{2}\left(\Gamma_{\lambda\mu}^{\nu} - \Gamma_{\mu\lambda}^{\nu}\right),$$
$$Q_{\mu}^{\cdot\lambda\nu} = \nabla_{\mu}\, g^{\lambda\nu}.$$

Then it is found that the displacement can be expressed in terms of C, S, Q, g, and the various kinds are classified according to the restrictions of C, S, Q. The affine displacement is characterized by

$$C = 0, \quad S' = \frac{1}{2}\left(\Gamma_{\cdot\lambda\mu}^{\prime\nu} - \Gamma_{\mu\lambda}^{\prime\nu}\right) = 0, \quad Q' = \nabla_{\mu}\, g_{\lambda\nu} = 0;$$

the Weyl displacement by

$$C = 0, \quad S' = 0, \quad Q = Q_{\mu}\, g_{\lambda\nu},$$

where Q_{μ} represents a vector; the Riemann displacement (defining the ordinary differential geometry) by

$$C = 0, \quad S' = 0, \quad Q' = 0.$$

For the general displacement a geodesic line is defined as a curve whose line elements are moved into line elements of the same curve and the differential equations of such a curve are

$$\frac{d^2 x^{\nu}}{d t^2} + \Gamma_{\lambda\nu}^{\nu}\frac{d x^{\lambda}}{d t}\,\frac{d x^{\mu}}{d t} = \alpha\frac{d x^{\nu}}{d t},$$

where α is a function of the coordinates. These have a marked similarity to the equations of geodesics in Riemannian geometry; and, in case $C = S' = Q' = 0$, they are identical with them.

Curvature is obtained by integrating δv^{ν} around an infinitesimal closed curve. The formula turns out to be very similar to that obtained by Levi-Civita for his parallel displacement. After fully developing these ideas Schouten then specializes the various kinds of displacements, devoting a chapter to each of the three mentioned above.

The whole structure is built up around the general idea of parallel displacement and cannot fail to convince the reader of the far-reaching importance of this notion. No geometer, unfamiliar with this subject, can afford to miss reading this book.

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