values between the extreme values of the function, and also that the same property holds for the sums of order two of the Laplace series. Considering also fractional values of the order k of the sums, the present paper proves that $k = 1$ and $k = 2$, respectively, are the smallest values of k with the above property.

27. Professor J. R. Kline: *A note concerning closed non-dense linear sets which are enumerable.*

In both the first and the second editions of Hobson's *Theory of the Functions of a Real Variable*, the author attempts to prove that a non-dense closed set is enumerable if its complementary intervals are such that every one abuts on another at each of its ends. Professor R. L. Moore gave an example of a set satisfying the conditions of the hypothesis of Hobson's theorem, which is not enumerable. In the present paper it is shown that in case it is stipulated that the non-dense closed set and all its derived sets have the property of having their complementary intervals each abut on another at each of its ends, then the set is enumerable.

R. G. D. Richardson,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION

The forty-fourth regular meeting of the San Francisco Section was held at the University of California on October 25, 1924. In the absence of the Chairman of the Section, Professor Carpenter, the meeting was called to order by the Secretary of the Section. Professor Hedrick was elected temporary chairman. The total attendance was thirty, including the following twenty-five members of the Society:

Alderton, Allardice, Andrews, Bell, Bernstein, Blichfeldt, Growe, Haskell, E. R. Hedrick, Hotelling, Irwin, Lehmer, Sophia Levy, McCarty, McFarland, F. R. Morris, Moreno, Noble, Pehrson, T. M. Putnam, Robertson, Schmiedel, Pauline Sperry, A. R. Williams, Wong.

The following officers were elected for the coming year: Chairman, Professor E. R. Hedrick; Secretary, Professor B. A. Bernstein; Program Committee, Professors E. T. Bell, H. F. Blichfeldt, B. A. Bernstein.

It was decided to hold regular meetings of the Section during 1925 as follows: at Stanford University on April 4, at the University of Oregon on June 19, and at the University of California in the fall, the exact dates and places to be determined by the officers of the Section.

Titles and abstracts of papers read at this meeting follow. In the absence of the author, Professor Bateman's paper was read by Mr. H. P. Robertson. Mr. Reimer was introduced by Professor Bernstein.

1. Professor Harry Bateman: *Numerical solution of an integral equation.*

Hafen has shown that the distribution of electricity on the circular plates of a parallel plate condenser can be found by solving a linear integral equation of the second kind. In the present paper the author solves this equation numerically by a method which is in some respects analogous to the method of least squares, and discusses the method in a general way, remarking on the question of convergence.

2. Professor E. T. Bell: *An algebra with singular zero.*

This algebra, which is that of the symbolic calculus for dealing with relations between the individual terms of sequences, is abstractly identical with common algebra, when properly interpreted, except that zero, defined by $a-a=0$, where a is any element of the algebra, has singular properties, such as $b+a-a\neq b$ (b an element of the algebra), $b(a-a)\neq 0$. An outline is given of the applications of this powerful tool for investigating the properties of numbers defined by sequences, also of the Bessel coefficients.

3. Professor E. T. Bell: *On certain functions of two variables and their integrals related to the Bessel coefficients.*

The functions $L_n(\lambda, \mu)$ in question are the coefficients of $t^n/n!$ ($n = 0, \pm 1, \pm 2, \cdots$) in the Laurent expansion, absolutely convergent for $|t| \neq 0$, of exp $(\lambda t + \mu t^{-1})$. By taking the coefficients of $t^n/n!$, and not of t^n, it is feasible to apply the umbral calculus of Blissard to the study of these functions and their integrals. For $\lambda = -\mu = \frac{1}{2}z$ the functions become the Bessel coefficients $J_n(z)$. Thus the relations between the $J_n(z)$ and the integrals dependent on them are obtained here in special cases. The L_n are readily expressed in terms of the J_n for irrational arguments. Many of the integrals thus evaluated in terms of Bessel coefficients appear to be new. The symbolic method applies equally well to the functions defined by exp $(F(t))$, where $F(t)$ is any finite or infinite polynomial in t, t^{-1}. These also are expressible in terms of Bessel coefficients. The use of the symbolic method cuts the computations to a minimum.

4. Professor E. T. Bell: *Modular equations and quadratic forms.*

The elliptic modular equation associated with the transformation of order n implies and is implied by each of several elementary arithmetical theorems concerning the representations of integers in certain quadratic forms with integer coefficients, of which at least one is divisible by n. For $n = 7$, in many respects the simplest case, there are 256 associated theorems, of which 4 refer to binary forms, 56 to ternary, and 196 to quaternary. These are given in full, in a condensed form.

5. Professor E. T. Bell: *On generalizations of the Bernoullian functions and numbers.*

All generalizations are segregated into two classes, (1) those obtained by modifications of the difference equations of definition, (2) those derived from modifications of the symbolic equations of definition. An example of (1) is Nörlund's memoir in ACTA MATHEMATICA, vol. 43, pp. 121–196. Here (2) only is discussed and illustrated by an extensive generalization of the "ultra-Bernoullian functions" of Krause (LEIPZIGER BERICHTE, vol. 55, pp. 39–63) and Appell (ARCHIV FÜR MATHEMATIK UND PHYSIK, vol. 4 (1902), pp. 292–5). The number of attainable generalizations is infinite. Suggestions are given for classifying this infinity.

6. Professor E. T. Bell: *Transformations of relations between numerical functions.*

From consideration of functions whose arguments are the elements of an abelian semi group, Cesàro (ANNALI DI MATEMATICA, (2), vol. 13, p. 155, (12), (13)) obtained a remarkable inversion of series, "le plus général que l'on connaisse," and states that the well known Möbius inversion is a special case. It is here shown that either one of the Cesàro or Möbius inversions implies the other, and that both are contained in an extremely general theorem of reciprocity, of which the Cesàro-Möbius inversion is the simplest example possible.

7. Professor E. T. Bell: *A new type of class number relations.*

The novelty is that the sums of class number functions are expressible as functions of the real divisors alone of a single integer when and only when the fixed integer in the arguments of the class number functions is a square. This paper appears in the October, 1924, number of the TRANSACTIONS.

8. Professor B. A. Bernstein: *A general theory of representation of finite operations and relations.*

At the International Mathematical Congress at Toronto, the author presented a method of representing arithmetically any abstract binary operation and any dyadic relation in a finite class. The method is now extended to the case of operations other than binary and relations other than dyadic. The representations bring out clearly the nature of an operation or relation.

9. Professor Florian Cajori: *Fanciful hypotheses on the origin of the forms of the Arabic numerals.*

The author describes numerous attempts, which have been made during the past nine centuries, to explain the origin of the forms of our nine numerals and the zero from the composition of short lines, or of angles, or of dots, and points out that all the hypotheses advanced proved fruitless because they failed to coordinate the known facts relating to the forms of the numerals at different times and in different countries, and failed to suggest useful new lines of further research.

10. Professor M. W. Haskell: *Note on the self-dual quintics.*

At the Mathematical Congress at Toronto, the author developed a method for deriving an unlimited number of curves which are autopolar with respect to a finite number of conics. In the present paper he exhibits two such curves of the fifth order, viz. (*a*) The curve whose equation in polar coordinates is $8\varrho^5 \cos 5\varphi - 15\varrho^4 + 10\varrho^2 - 3 = 0$ is a quintic with five cusps, invariant under a dihedral group of 10 linear substitutions, and hence autopolar with respect to six conics; (*b*) The curve whose equation is $(9\varrho^5 - 5\varrho^3) \cos 3\varphi - 15\varrho^4 + 15\varrho^2 - 4 = 0$ is a quintic with three cusps and three double points, invariant under a dihedral group of 6 linear substitutions, and hence autopolar with respect to four conics.

11. Professors E. R. Hedrick and Louis Ingold: *On certain attempted extensions of the theory of functions to three dimensions.*

The authors discuss attempts made by themselves and by others to extend the ordinary theory of functions of a complex variable to three dimensions by means of definitions of the product or of the quotient of two three-dimensional vectors. The nature of the difficulties encountered and the reasons for them are pointed out. Another road toward extension of the theory was finally adopted, and has been presented to the Society (see this BULLETIN, vol. 29, p. 437), but the reasons for abandoning the more obvious path have not been stated, and it seems not without interest to state them.

12. Dr. Harold Hotelling: *Three-dimensional manifolds of states of motion.*

To every two-dimensional dynamical problem with constant total energy there corresponds a three-dimensional manifold, two coordinates determining the position of the moving particle on the surface given by the problem, and the third the direction of its motion. The topology of this manifold, which is important in the study of periodic orbits, depends only upon that of the portion of the given surface on which the particle may move. Characterizations of all of these manifolds and formulas for their principal invariants are given.

13. Dr. F. R. Morris: *Development of annuity formulas without series.*

The formula $s = (1+r)^n$ may be taken as the definition of the amount on one unit with interest at rate r per period for n periods. Then $(1+r)^n - 1$ is the interest on one unit. $(1+r)^n - 1$ is also the value of an annuity with periodic rent r. The author makes use of these facts to derive the well known annuity formulas without the help of series.

14. Mr. E. H. Reimer: *Application of Weyl's geometry to differential geometry in two dimensions.* Preliminary report.

If neighboring vectors of zero length are equivalent, the geometries of Weyl and Eddington coincide for the surface $S(u_1, u_2)$. For Eddington's $\lambda g_{ij} = R_{ij}$, the Gauss equations give $\lambda = R$. Either of two vectors $V_i(u_1, u_2)$ $(i = 1, 2)$ in the asymptotic directions of length $\sqrt{R}$ gives a Weyl vector. Accents denoting covariant derivatives, the Codazzi equations become $V_1' = \pm V_2'$, according as the parameters are (u_1, u_2) or (u_2, u_1).

15. Mr. H. P. Robertson: *On certain solutions of the cosmological equations.*

This paper concerns itself with certain solutions of Einstein's equations in a space-time which may be described by orthogonal coordinates. A solution is found which gives the field due to a line or plane distribution of matter and electricity. It is proved that certain types of line element lead only to the central-symmetric field of Schwarzschild. Other solutions are found which give rise to theorems of interest in differential geometry and the new physics; in these cases simple data of observation are the criteria of the form of space-time.

B. A. Bernstein,
Secretary of the Section.