THE SCIENTIFIC WORK OF JOSEPH LIPKA

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Joseph Lipka, Associate Professor of Mathematics at the Massachusetts Institute of Technology, died January 15, 1924, in his forty-first year and the sixteenth year of his service at the Institute. He was a mathematician of proven worth, intensely interested in his science, and unremitting in his efforts to forward its progress. The last few years of his life were especially remarkable for their productivity. During this time he contributed what would normally be considered, both in volume and value, as the creditable work of a decade, and gave abundant promise of continued fruitfulness. That he has been taken in his prime is indeed a source of great regret and a distinct loss to mathematics.

During Lipka's student days at Columbia, Professor Kasner was busily applying differential geometry to dynamics and developing, in euclidean spaces of two and three dimensions, the geometric properties of dynamical trajectories and related systems of curves. It was this field of investigation in which Lipka was later initiated and did his work for the doctorate. His thesis consisted in a generalization, to a curved space of n dimensions, of results obtained by Kasner concerning a natural family of curves, that is, a family of trajectories in a conservative field of force corresponding to a given constant of energy, or more generally, any family of extremals resulting from a problem in the calculus of variations of the form,

$$\int e^{\varphi} ds = \text{minimum}, \tag{1}$$

where ds is the linear element of the space in question, and φ is a function of the coordinates of the space.

Lipka's later work, though largely confined to the one field, falls naturally into two periods. The earlier of these sees the generalization of many of Kasner's theories, first to ordinary surfaces and then to curved spaces of n dimensions. Noteworthy is the exhaustive study of dynamical trajectories on a surface for any positional field of force, and also the investigation of the geometrical properties of trajectory and related systems in Riemannian n-space. But perhaps the most important achievement of this period was the paper which established the validity, in all cases, of the Thomson-Tait criterion for a natural family. According to the theorem of Thomson and Tait, the ∞^{n-1} curves of a natural family meeting an arbitrary hypersurface orthogonally form a normal hypercongruence, i. e., admit of ∞^1 normal hypersurfaces. That this property is characteristic of natural familes (for $n > 2$) had been proved by Kasner for a euclidean space of three dimensions. In the paper now under discussion, this was proved for a euclidean space, first of four and then of n dimensions, and finally for a general curved space.

During his career Lipka made two trips abroad, and both of them he turned to good account. The first, during the summer of 1913, brought him into contact with Professor Whittaker and the mathematical laboratory at Edinburgh and thus furnished the inspiration which led him to found at the Institute a laboratory course and later to write the text on *Graphical and Mechanical Computation.*

The summer of 1921 and the following academic year Lipka spent on leave in Europe, principally in Rome. This sojourn abroad marked the beginning of a new period in his scientific work. The stimulus and inspiration which he derived from his new associations, especially from those with Professor Levi-Civita, are clearly evident in his writings. His work is now permeated with the methods of the absolute calculus and governed largely by new ideas. As a first illustration may be mentioned the use of Hamilton's canonical equations and infinitesimal contact transformations to prove anew, in a brief and elegant fashion, the Thomson-Tait characteristic property of natural families, and to extend this property so that it applies to an irreversible system; in the extension, which of course applies merely to a conservative field of force, it is no longer the curve which is orthogonal to the hypersurfaces in question, but the resultant of the velocity vector along the curve and the vector whose components are the coefficients of the (new) linear term in the Lagrangian function.

Another group of papers revolve about Levi-Civita's notion of parallelism, or parallel motion of a direction along a curve, in a Riemannian space. By use of this concept Lipka gives a simple intrinsic definition of geodesic curvature, which is the analogue of that of ordinary curvature in euclidean space. Thus, if P' is a point on a curve C neighboring to a given point P of C, and if $d\omega$ is the angle between the tangent at P' and the parallel at P' to the tangent at P, the limit of the ratio of $d\omega$ to the arc PP' is the geodesic curvature of C at P. In a later paper a similar process is employed to obtain the curvature of C at P relative to any curve C' tangent at P to C. This new type of relative curvature finds application in dynamics when the curve C' is taken as the trajectory of a natural family.

From equation (1) it is clear that natural families are a generalization of geodesics and that every natural family is the conformal representation of the geodesics in a second space. It was doubtless these facts which led Lipka to generalize Levi-Civita's concept of parallelism by replacing the geodesics at the basis of it by the trajectories of a natural family, and which suggested for the new type of parallelism the name "conformal parallelism". This idea, by its very nature, is of value in the study of dynamical problems. For example, a trajectory is now characterized by the fact that its direction at any point is conformally parallel to its initial direction. Again, the curves orthogonal to the ∞^1 trajectories on an ordinary surface which issue at right angles from a given curve are conformally parallel.

Perhaps the aspect of conformal parallelism which will make the widest appeal is its close connection with the second differential parameter. Levi-Civita has shown that the Gaussian curvature of a surface at a point P is the ratio of the angle between the initial and final positions of a direction which moves by parallelism completely around an infinitesimal cycle through P, to the area of the cycle. Lipka shows that, in the case of conformal parallelism, this ratio is equal to the difference between the curvature at P and the value at P of $\Delta_2\varphi$, where φ is the "characteristic" function given in (1). If, then, φ is any point function on the surface, the value at P of $\Delta_2\varphi$ can be interpreted as the ratio of the angle between the two final positions of a direction which moves first by parallelism and then by conformal parallelism (with the characteristic function φ) completely around the cycle, to the area of the cycle.

The substitution of trajectories for geodesics and trajectory surfaces for geodesic surfaces forms also the basis for the development, in another paper, of a generalization of the Riemannian theory. Thereby Riemannian curvature is replaced by trajectory curvature, that is, the Gaussian curvature of a trajectory surface at a point. Moreover, the theory developed by Ricci concerning principal directions and principal curvatures lends itself to complete generalization. The new theory contains the old as a special case, when φ is constant. The relationships between the fundamental quantities in the two theories are thus readily obtained and are quite remarkable in their simplicity and elegance.

Lipka's last published paper has to do with a system of invariants introduced by Ricci in his study of orthogonal congruences, and called by him coefficients of rotation, in light of the fact that he was able to show that they can be interpreted in terms of rotations in the euclidean space in which the given curved space is imbedded. Following a suggestion of Levi-Civita, Lipka proves that these invariants admit a simple intrinsic interpretation within the given space by means of the notion of parallelism. This interpretation enables him to give elegant characterizations of certain types of congruences, notably congruences of geodesics, in terms of the vanishing of certain of the invariants, and to obtain other results of equal importance. Finally, by introducing conformal parallelism instead of that of Levi-Civita, he develops a new set of invariants, which yield corresponding results bearing on trajectories.

It is a pleasure to read Lipka's writings. The habit of clear thinking and clear expression was ingrained in him. To this, as well as to his personal qualities, was due his success as a teacher.

Lipka was natural, unpretentious, always cheerful and sympathetic, ever willing to lend a hand, and enthusiastically devoted to his chosen work. And beneath these traits there was a definite, but more elusive, quality which vitalized them, a certain inner buoyancy, an ever forward pressing optimism, an intangible force which drew others to him.

LIST OF LIPKA'S WRITINGS

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