WINGER ON PROJECTIVE GEOMETRY

An Introduction to Projective Geometry. By R. M. Winger. Boston, D. C. Heath, 1923. V + 443 pp.

Among the books on projective geometry in English which have appeared so far, Winger's *Introduction* is a distinct novelty. In fact it justly breaks away from the more or less traditional Cremona-Reye style of a passed period and presents, on the whole, such topics which for a more advanced study of geometry are of essential importance. For this reason, the reviewer is glad to declare from the start that Winger has written an excellent text-book.

The author himself says that "this book is intended as an introductory account for senior-college and beginning graduate students — for the prospective teacher who is seeking proper orientation of elementary mathematics, as well as the university student who lacks the preparation for an intelligent reading of the general treatises on higher geometry and the modern books on higher algebra". As mathematical preparation for a proper understanding of the book collegiate training in algebra, analytic geometry and calculus is all that is required. It is perhaps not necessary to review in detail the thirteen chapters which in order deal with essential constants; duality; the line at infinity; projective properties; double ratio; projective coordinates; the conic; collineations and involutions in one dimension; binary forms; algebraic invariants; analytic treatment of the conic; collineations in the plane; cubic involutions and the rational cubic curve; non-euclidean geometry. The book is very clearly and expressively written throughout, and the propositions are stated concisely and in simple straightforward English. Undoubtedly it will have a very refreshing effect upon the student.

The few criticisms which the reviewer wishes to make concern in some instances the method of presentation rather than the choice of contents. Thus, on page 8, the statement "to show that a condition is necessary and sufficient entails the proof of a proposition and its converse" is, of course, not obvious and needs qualification. In the definition of isotropic lines by $x \pm iy - k = 0$, it would probably be better not to introduce the new term *circular rays* for the special case when $k = 0$. The old term *ray* (*Strahl*) which has an optical meaning when qualified should be abandoned. The word *line* (straight) is sufficient for this purpose. Thus one might speak of $x \pm iy = 0$ as the principal isotropic lines. The paradoxical statements at the top of page 54 might have been omitted without harm; since from a purely geometrical standpoint they are absolutely meaningless.

In Chapter IV on projective properties the statement that "between a figure and its projection there exists the relation of *one-to-one-correspondence*" should be qualified as *algebraic*, in order to be projective, since it is not difficult to establish non-projective one-to-one-correspondences, as for example between the x- and y-axis of a cartesian system by means of $y = \sqrt[3]{x}$ when restricted to real numbers.

Discussing metric and projective properties the author says that "since a circular cone is merely a perspection of a circle from the vertex it follows that any section, not on the vertex; i. e., any proper conic is a projection of the circle". Now there is certainly no obvious conclusion impelled by such an assumption. No matter whether the spaces in which one operates are metrically or projectively defined, such a statement needs proof.

The treatment of projective coordinates is in the opinion of the reviewer the least satisfactory in the whole treatise. The term projective applied to any mathematical doctrine means independence of any particular metrical concept whatsoever. Hence from a scientific standpoint, the establishment of projective coordinates should not be based either on euclidean or cartesian space, even when disguised in the form of a homogeneously made cartesian space. The proper starting point is the famous, indefinitely repeated harmonic quadrangle construction of v. Staudt, plus Dedekind's axiom of continuity, etc., and the establishment of a linear (1,1)-correspondence between the elements thus obtained and the continuum of real numbers. Thus on a line every point A_i is individualised by a parameter $\lambda_i = x_1/x_2$. Projectively, the distance between two points A_1 and A_2 is defined by $A_1 A_2 = \lambda_2 - \lambda_1$. A similar procedure leads to the definition of projective coordinates of spaces of two, three, or, in fact, of any number of dimensions. We easily obtain more general spaces by admitting complex values for the parameters λ. In this manner, a perfectly rigorous projective geometry may be built up without the least reference to euclidean or cartesian metrics. Also from a didactic standpoint this procedure is simple enough and does not offer greater difficulties to the student than other more or less metric methods. In a purely projective geometry (of the plane) the statement that "the projective coordinates of a point are proportional to fixed multiples of the distances of the point from the sides of the triangle of reference" has no place at all.

The chapters on binary forms and algebraic invariants are very well done. In courses on the algebraic theory of invariants the greater part of the time is usually spent in the study and assimilation of the purely formal algebraic processes, so that little time is left for geometric applications. Winger's presentation of the subject, based on the polarizing process, may be called ideal from the standpoint of the geometrician. The treatment of apolarity in the ternary field, however, must seem rather abrupt and disconnected to the student, since it is based merely on a suggested generalization from the binary field.

"Absolute coordinates," defined by

$$x = X + iY, \overline{x} = X - iY,$$

might better be called *isotropic coordinates*, as introduced by Laguerre. If one speaks of isotropic lines it would seem natural to speak also of isotropic coordinates. The term *absolute* introduced by Cayley is not expressive of the idea it intends to convey. There certainly is nothing "absolute" about such real or imaginary boundary conics of non-euclidean geometry.

The chapter on collineations brings out the important properties of these transformations. Involutions on the rational cubic curve are treated in a very instructive manner. The idea of order and class of a curve resulting from the principle of duality is illustrated effectively by a number of well chosen examples. The restriction of involutive properties to rational curves reflects, of course, the author's preoccupation with such curves in his researches on self-projective rational curves. On the other hand, the elliptic cubic, as is well known, offers a beautiful example for the application of projective theories.

In the concluding chapter we find a short account of non-euclidean geometry in the plane after the style of Klein, and a historic note thereupon. That a historic account in a book on projective geometry should be limited to non-euclidean geometry seems somewhat peculiar. It would have been perfectly proper to give a general historic sketch on the entire subject of projective geometry.

It is proper to point out an error which is very common in histories of mathematics and which is contained in the following statement on page 420. "Little progress was made until about a hundred years later when Gauss (1777-1855) and his friends and pupils became deeply interested in the subject." Now the fact is that Gauss's deeper interest in the subject was subsequently aroused by the brilliant discoveries of Lobatschewsky and Bolyai. As a matter of fact, Gauss, in the beginning, hoped to be able to prove what is known as the euclidean parallel axiom and assumed a rather skeptical attitude towards the new discoveries. Subsequent deeper meditations, however, led Gauss to his own establishment or verification and acceptance of the new theory.

A very valuable feature of Winger's introduction is the number of well chosen exercises. Altogether the book will serve a very useful purpose. It will enable the student to acquire the basic concepts and theories which are nowadays a necessity for students who wish to enter into the study of more advanced geometric theories. In this sense the book deserves to be very highly recommended.

ARNOLD EMCH