THREE BOOKS ON RELATIVITY

The Mathematical Theory of Relativity. By A. S. Eddington. Cambridge, University Press, 1923. VI + 247 pp.

The Meaning of Relativity. By Albert Einstein. Translated into English by E. P. Adams. Princeton University Press, 1923. 123 pp.

The Theory of General Relativity and Gravitation. By L. Silberstein. New York, Van Nostrand, 1922. IV + 141 pp.

In attempting a combined review of these three books, I do not wish to give the impression that each does not merit extensive consideration. However, there are necessarily many points of similarity, and by pointing these out and emphasizing the differences I may be able to give an idea of the character of each book. Eddington's book is by far the most comprehensive and contains practically all of the mathematical treatment appearing in the other books, but the latter contain many helpful and stimulating observations and interpretations.

Einstein recalls in his first lecture some of the more fundamental ideas and equations of pre-relativity physics, and converts them into tensor form. In the second lecture the equations for the same physical concepts as interpreted in special relativity are given tensor form. This should be particularly helpful to those, who, not being entirely familiar with mathematical processes, have tried to acquire a knowledge of tensor analysis from the general treatments, such as was given by Einstein in his 1916 paper, and is largely followed in the books under discussion (Eddington, Chapter 2; Einstein, Lecture 3; Silberstein, Chapter 3). After following the first two lectures the reader begins to feel that, as Eddington says (3), "our knowledge of conditions in the external world, as it comes to us through observation and experiment, is precisely of the kind which can be expressed by a tensor and not otherwise." He may not yet be prepared to agree with the last part of this statement, but as he comes to appreciate the effective use which Einstein has made of tensor calculus in his general theory of relativity, he is forced to the conclusion that here is a great contribution to mathematical physics. Scientists may agree or not with Einstein's interpretation of his equations as regards the character of physical space and, in particular, the significance of the well known crucial tests of his theory, but they cannot afford to ignore the guidance of tensor calculus in their attempts to give mathematical formulation to the results of experiment.

The postulates and ideas of special relativity are set forth, more or less briefly, by all three authors in preparation for the transition to general relativity. In their generalized form the postulates may be stated in the explicit form:

1. Everything connected with location which enters into observational knowledge—everything we can know about the configuration of events—is contained in a relation of extension between pairs of events; this relation is called the interval and its measure is denoted by ds; in any system of coordinates $ds^2 = \Sigma g_{ij}\, dx^i\, dx^j$, where g_{ij} are functions of the coordinates which describe the metrical relations of the space-time continuum and also the gravitational field.
2. The path of a freely moving particle, whether in the presence or absence of a gravitational field, but not in an electromagnetic field, is a geodesic of the quadratic form ds^2.
3. The track of a light-wave satisfies the condition $ds = 0$. (Ed. 10, 36; S. 20; E. 71, 87, 102.)

The generalization consists in replacing inertial, or Galilean, systems by general systems of coordinates and in interpreting the g's as potentials of the gravitational field. As a first step in this and other generalizations, we have Einstein's postulate:

4. For infinitely small regions there will be an inertial system (i. e. a local reference system for which the g's have the Galilean values) relative to which the laws of special relativity are valid. (E. 70; S. 12.)

However, the assumption underlying the generalization is Einstein's "principle of equivalence". Eddington (40) considers the latter to be "a hypothesis to be tested by experiment as opportunity offers"; ... "as a suggestion rather than a dogma admitting of no exceptions." Under the guidance of this principle Einstein was led to the postulates:

5. Inertial mass and gravitational mass are identical.
6. The law of gravitation for empty space is $R_{ij} = 0$, or $R_{ij} = \lambda g_{ij}$, where λ is a very small constant. It may be that this is equivalent to the "principle of equivalence," in other words, that the principle breaks down for all other curved worlds.

Proceeding from these postulates, Eddington (83–90) and Silberstein (92–100) derive the Schwarzschild solution of the equations $R_{ij} = 0$ for a radially symmetric field and obtain the equations of the geodesics which are identified with the paths of the planets about the sun; from these are derived Einstein's formula for the motion of the perihelion of Mercury; Einstein (105–07) merely states these results. Eddington adds to Postulate 3 the requirement that the tracks of light be geodesics; and, by allowing ds to approach zero in the preceding discussion, he obtains the formula for the deflection of light; Silberstein gives a similar derivation and also states that it is a consequence of Fermat's principle, as shown by Levi-Civita and de Sitter. Einstein does not assume that the paths of light are geodesics. He adheres to Postulate 3 and derives (103) the formula for deviation of light from an approximate solution of the equations

$$R_{ij} - \tfrac{1}{2} g_{ij} R = -kT_{ij}. \tag{1}$$

From this solution he determines the value of the constant k and upon it he bases also his conclusion concerning the shift of spectral lines; the discussion of this question by Eddington and Silberstein is based on the expression $ds^2 = (1 - 2m/r)\,dt^2$, obtained from the Schwarzschild form for an atom at rest or practically so.

Before giving the results for planetary motion as stated above, Einstein (88) considers an approximate solution somewhat after the manner of his 1916 paper. Silberstein (35–8) follows a similar course and obtains the equations given by Einstein with added terms. He thinks that Einstein failed to obtain these "through a too-hasty computation of the Christoffel symbols." It seems to be a question rather of what is meant by "first approximation." As interpreted by Einstein in both places these added terms vanish. In any case the interpretation of these terms as due to an acceleration exerted upon the frame of reference by a velocity field cg_{4i} $(i = 1, \ldots, 4)$ is interesting.

Einstein's derivation (90–94) of equations (1), as a generalization of the Laplace-Poisson equation,

$$\Delta\varphi = 4k\pi\rho, \tag{2}$$

is one of the most interesting parts of the book. The tensor T_{ij}, the energy-momentum tensor of matter, "includes in itself the energy density of the electromagnetic field and ponderable matter", ... "It is only the circumstance that we have not sufficient knowledge of the electromagnetic field of concentrated charges that compels us, provisionally, to leave undetermined in presenting the theory the true form of this tensor." However, in accordance with special relativity, the principle of conservation of momentum and energy is expressed by the vanishing of the divergence, that is

$$\frac{\partial T_{ij}}{\partial x^i} = 0. \tag{3}$$

This and Postulate 4 serve as a guide in the determination of the left-hand member of (1). Einstein does not say that this is equivalent to

$$T^{ij}_{i} = 0 \tag{4}$$

in general coordinates, where the subscript indicates covariant differentiation, but he explicitly states (91) "we shall have to assume (4) as valid." We wish to emphasize that there is an assumption involved in replacing ordinary derivatives of special relativity by covariant derivatives in general relativity—it is the assumption that in applying Postulate 4 the coordinates at the point are geodesic. Eddington (119) makes the transition from (3) to (4) by an appeal to the "principle of equivalence." Since the components g_{ij} are interpreted as the potentials of the gravitational field—a fundamental assumption of the Einstein theory—the generalization of equation (2) requires of the left-hand member of (1) that it contain no differential coefficients of g_{ij}

higher than the second, that it be linear and homogeneous in the second differential coefficient, and that its divergence be zero. Einstein states without proof that the only tensor of the second order satisfying the first two conditions is $R_{ij} + a g_{ij} R$, in the usual notation, where a is an arbitrary constant.* He then shows that the third condition requires that $a = -1/2$, and in doing so makes use of a particular type of geodesic coordinates at a point. In view of his assumption mentioned above, he is then justified in his statement that the vanishing of the divergence is proved for any system of coordinates. Eddington (119) and Silberstein (81) establish the same result by making use of the four fundamental identities

$$R^i_{ij} = \tfrac{1}{2} \frac{\partial R}{\partial x^j}.$$

So far as I know, these identities were first established by Levi-Civita (RENDICONTI LINCEI, 1917, p. 388) in a general way by means of the identities of Bianchi; Eddington's proof (115) is based upon the use of geodesic coordinates.

The tensor T_{ij} for an electromagnetic field alone was written by Einstein in his 1916 paper (§ 20) in the form

$$T^i_j = -F_{j\alpha} F^{i\alpha} + \tfrac{1}{4} \delta^i_j F_{\alpha\beta} F^{\alpha\beta}, \tag{5}$$

where $F_{j\alpha}$ is the curl $\partial \varphi_j / \partial x^\alpha - \partial \varphi_\alpha / \partial x^j$ of the electromagnetic potential, δ^i_j are the Kronecker deltas, and

$$F^{\alpha\beta} = g^{\alpha i} g^{\beta j} F_{ij},$$

using the conventional notation. In his second lecture (53) he shows that this expresses the principles of energy and momentum as developed by Maxwell, and that the four-dimensional formulation of special relativity serves as the guide to the amalgamation into a tensor. By similar considerations Maxwell's equations are written in the tensor form

$$\left\{\begin{aligned} &\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0, \\ &\frac{\partial}{\partial x^j}(\sqrt{-g}\, F^{ij}) = \sqrt{-g}\, \rho_0 \frac{dx^i}{ds}, \quad \text{or} \quad (F^{ij})_j = \rho_0 \frac{dx^i}{ds}, \end{aligned}\right. \tag{6}$$

where ρ_0 is the proper density of electricity and $(F^{ij})_j$ is the covariant derivative. (Cf. Ed. 170–5; S. 106–113.) In this generalization from special relativity by means of Postulate 4, there is the added assumption that the coordinates are geodesic. Applying covariant differentiation to (5), we have (Ed. 182; S. 121) for the divergence of T^i

$$T^i_{ji} = F_{j\alpha} \rho_0 \frac{dx^\alpha}{ds}. \tag{7}$$

Thus the divergence vanishes in regions outside of charged particles,

* For a proof the reader is referred to Weyl, *Space, Time and Matter*, p. 315; also Birkhoff, *Relativity and Modern Physics*, pp. 209–221.

"the only regions in which we can believe that we have the complete expression for the energy tensor", according to Einstein (55). The expression in the right-hand member of (7) is the ponderomotive force four-vector which replaces the electric force vector in the equations of electrostatics, when these equations have been put in tensor form. These equations were derived for special relativity by Einstein in his 1905 paper and again in his second lecture (52). They agree to a high degree of accuracy with the results of experiments on β-particles; and, when written in the form for general relativity, they show that the paths of an electron are not geodesics. Eddington (189) "wants to know what the electron is trying to accomplish by deviating from a geodesic". He considers the field outside the electron and assumes that the field within the electron counterbalances it; he then concludes that an electron in an external field of force having the acceleration given by the equations referred to is "a miracle", whatever that may mean.

As previously remarked, Einstein obtained an approximate solution of equations (1) and therefrom reached the conclusions concerning physical phenomena which have become generally known. These solutions were based on the assumption that the potentials g_{ij} have the Galilean values at infinity and differ little from these values in the neighborhood of matter. In 1917 he presented his views on the so-called *cosmological problem*. Mach held that the inertia of any particle depends upon the whole matter in the universe. If this is true, then (1) "the inertia of a body must increase when ponderable masses are piled up in its neighborhood; (2) a body must experience an accelerating force when neighboring masses are accelerated, and, in fact, the force must be in the same direction as the acceleration; (3) a rotating hollow body must generate inside itself a 'Coriolis field', which deflects moving bodies in the sense of a rotation, and a radial centrifugal field as well" (E. 110). Einstein obtained an approximate solution, involving each of these effects, but of magnitude too small to be tested by experiment; in doing so he was led to make the hypothesis that the physical universe, as distinguished from the space-time continuum, is spherical and closed, and he proposed suitable expressions for the functions g_{ij} and for the tensor T_{ij} which were in keeping with this hypothesis. We cannot here discuss adequately the merits and demerits of this proposal, and of the one suggested by de Sitter; the reader will find this done fully by Eddington (155–168) and by Silberstein (124–137).

Suppose that we had not started with the six postulates previously set down, but had merely taken the first one, which may be interpreted roughly as saying that the physical world (space-time) is a Riemannian manifold of four dimensions with a fundamental tensor g_{ij}. From this tensor others may be derived, such as R_{ij}, B_{ijkl}, and so on. Accepting the fundamental principle of relativity that physical laws are expressible by tensor equations, the next step is to find these equations by a principle of identification with the results of experiment. If, as Eddington (119) says, we take "the view that energy, stress and momentum belong to the world and not to some extraneous substance in the world, we must identify the energy-tensor with some fundamental tensor of the world". Since the divergence of $R_{ij} - \frac{1}{2} g_{ij} R$ vanishes, it is natural to try the equation (1) and see what happens. The left-hand member vanishes only when $R_{ij} = 0$; this, then, is the condition for empty space. From (1) it can be shown that the path of a particle in empty space is a geodesic (Ed. 127). By identifying geodesic coordinates at a point with the Galilean coordinates of special relativity, Eddington (178) shows that the path of a light-pulse is a geodesic; it is a question, however, whether this proof is not equivalent to the assumption involved in the identification. We shall not consider the further consequences of this point of view, but will turn our attention to the more general world-geometry, as developed by Weyl and Eddington.

Weyl and Eddington consider a general four-dimensional continuum in the sense of Analysis Situs and define an affine connection for the manifold by means of forty functions Γ^i_{jk}, symmetric in j and k. By considering the change in a general vector A^i as it undergoes a parallel displacement round a small circuit, they are led to a tensor of the fourth order $^*B^i_{jkl}$ which is a generalization of the Riemann tensor. When this tensor is contracted for i and l, the resulting tensor is the sum of a symmetric tensor R_{jk} and the curl of a vector φ_i. Eddington puts $R_{jk} = \lambda g_{jk}$, where λ is a universal constant and takes

$$ds^2 = g_{ij}\, dx^i dx^j$$

as the metric of the space; he leaves the functions Γ^i_{jk} perfectly general, which is equivalent to having an arbitrary tensor K^i_{jk}, symmetric in j and k. Weyl, on the other hand, specializes his geometry by expressing the Γ's as certain functions of the components of a symmetric tensor g_{ij}, their first derivatives, and the components of a vector, which is equivalent to φ_i mentioned above.*

Before we proceed further with the development of this geometry

* Eddington's world-geometry is essentially the same as the Geometry of Paths as developed by Veblen and myself in a number of articles in volumes eight and nine of the PROCEEDINGS OF THE NATIONAL ACADEMY, and by me in the ANNALS OF MATHEMATICS, vol. 24 (1923). This geometry is not equivalent to the more restricted type considered by Weyl, as Eddington states (243). Furthermore, the equations of the paths are fundamental, which has a bearing on Eddington's remark (216): "It may be asked whether there is any other way of obtaining tensors. besides the consideration of parallel displacement round a closed circuit. I think not . . ."

it is advisable to consider its relation to the actual space of physical phenomena. Eddington (197) says "Two possible ways of generalizing our geometric outlook are open. It may be that the Riemannian geometry assigned to actual space is not exact; and that the true geometry is of a broader kind leaving room for the electromagnetic potential vector to play a fundamental part and so receive geometric recognition as one of the determining characters of actual space." ... "The alternative is to give all our variables, including the electromagnetic potential, a suitable graphical representation in some new conceptual space—not actual space." ... "We have then to distinguish between Natural Geometry, which is the single *true* geometry in the sense understood by the physicist, and World Geometry, which is the pure geometry applicable to a conceptual graphical representation of all the quantities concerned in physics." The latter is Eddington's idea of the significance of his geometry and of Weyl's; Weyl at first held that his geometry was the geometry of actual space, but according to Eddington (198, 208) he now holds to the latter view also. Commenting upon the theory of electromagnetism leading to equation (7), Einstein (108) says "This inclusion of the theory of electricity in the scheme of the general theory of relativity has been considered arbitrary and unsatisfactory by many theoreticians. Nor can we in this way conceive of the equilibrium of the electricity which constitutes the elementary electrically charged particles. A theory in which the gravitational field and the electromagnetic field enter as an essential unity would be much preferable." He reiterates this statement in his latest paper (PREUSSISCHE AKADEMIE DER WISSENSCHAFTEN 1923, pp. 32–38) in which he undertakes to present such a theory. However, he gives no indication that, in adopting Eddington's geometry for the basis of his work, he feels that the geometry is merely a graphical representation of the physical world. On the contrary the idea that he is dealing with the actual world seems to be fundamental in all of his writings concerning the theory of relativity.

Weyl, Eddington, and Einstein identify the vector φ_i of their geometries with the electromagnetic potential. Eddington (208) says it is "the electromagnetic potential because that is the way in which we choose to represent the potential graphically." He also identifies $R_{ij}\,dx^i\,dx^j = 0$ as the equation of light-pulses, but he does not prove that this is a consequence of the former identification. In other words, it seems to me that if Eddington's view of graphical representation is adopted and one identification is made, the other must be established. It seems reasonable that in some way φ_i is connected with the electromagnetic potential, but if one is dealing with actual space is it in fact the electromagnetic potential itself, or if one is making a graphical representation, is that the best interpretation to place upon φ_i? Recently I gave (PROCEEDINGS OF THE NATIONAL ACADEMY, June, 1923) another identification of φ_i based upon the equations of a moving electron. In the present state of the theory, a statement that this is the correct interpretation would be dogmatic. I mention it merely for the purpose of pointing out that if it is found that the conclusions of Weyl, Eddington, and Einstein are not in accordance with physical phenomena, as they become more clearly understood, the difficulty may be due to their identification of the vector φ_i.

It would not be proper to bring this review to a close without saying something about the Hamiltonian principle of action. Lorentz and Hilbert seem to have been the first to try to take over this principle from classical mechanics and determine its bearing upon general relativity. Einstein considers it in his 1916 paper (§ 15) and later in the same year made a more extensive study of it. Silberstein (88) feels that "the importance of the Hamiltonian principle seems to be unduly overestimated," and Eddington, who gives (131–144) a full discussion of it, together with what he calls the Hamilton derivative, says (138) "We have thus to remember that when a writer begins to talk about action, he is probably going to consider impossible conditions of the world. (That does not mean that he is talking nonsense—he brings out the important features of the possible conditions by comparing them with the impossible conditions)." There are two distinct methods of approaching this question. One is to build up an integral such that on equating its variation to zero, known results are obtained. This is what Weyl has done (*Space, Time, and Matter*, pp. 209–216; pp. 230–237). The other is to start with the idea that an integral can be found which will be the philosopher's stone to reveal the hidden treasures of the physical world. Thanks to the guiding hand of the tensor calculus the search for such an integral (if it exists) is not a game of blind man's buff. Weyl (295) chooses an integral and discusses the consequences of the choice, although he questions whether his action-principle is realized in nature exactly in the form chosen. Eddington (232) suggests and considers several integrals, one of which Einstein develops in his recent paper in such a manner as to obtain an expression for the Γ's in terms of a tensor s_{ij} and a vector i_j. The results of these theories await verification. It may be that one of them is the hoped-for *omnium gatherum* with or without the correct identification.

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