PROOF OF A FORMULA FOR AN AREA*

BY H. E. BRAY

1. *Introduction.* The formula

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)} dudv,$$

representing the area of the image of a (u, v) rectangle R in the (x, y) plane, can be proved to hold in the case where the transformation $x = x(u, v)$, $y = y(u, v)$ is of a very general kind.† The purpose of this paper is to extend and prove the formula by means of an approximating function of a very simple kind. The main properties of this function are given in § 3 and used in the proof of the formula, § 4.

2. *Definitions.* The approximating function for the summable function $f(x, y)$ is given by the formula

$$f^{(\mu)}(x, y) = \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} f(x + \xi, y + \eta) d\xi d\eta$$

where σ_μ represents the square region $[-\mu \leqq \xi \leqq \mu, -\mu \leqq \eta \leqq \mu]$, also the area of that square.‡ For convenience $f(x, y)$ is regarded as summable (Lebesgue) in the fundamental region S $[0 \leqq x \leqq 1, 0 \leqq y \leqq 1]$ and the properties of $f^{(\mu)}$ are considered with reference to a rectangle R $[a \leqq x \leqq b, c \leqq y \leqq d]$ inside S, μ being less than $a, c, 1 - b, 1 - d$.

The formula to be proved involves *generalized derivatives* and *potential functions*. These are defined as follows:

DEFINITION (i). If α is a given direction and α' the direction 90° in advance of α, the quantity

$$D_\alpha f(x, y) = \lim_{\sigma \to 0} \frac{1}{\sigma}\int_s f(x, y) d\alpha',$$

if it exists, is called the *generalized derivative of f in the direction α.* It is understood that the integral is taken in the positive

* Presented to the Society, December 27, 1922.

† W. H. Young, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 18, p. 339.

‡ It is convenient sometimes to use as σ_μ the interior of the circle $\xi^2 + \eta^2 = \mu^2$. The properties of $f^{(\mu)}$ are essentially the same in this case.

sense around the contour s, of a square σ, having its center at (x, y) and its sides parallel to the coordinate axes.§

DEFINITION (ii). Let $\varphi(x, y)$ be a vector point function whose component φ_α in every fixed direction, α, is summable superficially, and $u(x, y)$ a scalar point function, summable superficially, and such that $\int_s u(x, y)d\alpha$ may be defined for every given direction α. Then if φ and u are so related that $\int\int_\sigma \varphi_\alpha d\sigma = \int_s u d\alpha'$ for every rectangle σ* and for every fixed direction α, u is said to be a *potential function for the vector* φ.

3. *Properties of the Function* $f^{(\mu)}$.†

(1) If $f(x, y)$ is summable superficially $f^\mu(x, y)$ is a continuous function of (x, y), absolutely continuous in x for every y and in y for every x. At nearly every point of any line $y =$ const. $\partial f^{(\mu)}/\partial x$ exists:

$$\frac{\partial}{\partial x}f^{(\mu)} = \frac{1}{\sigma_\mu}\int_{s_\mu} f(x + \xi, y + \eta)d\eta,$$

the integral being evaluated about the contour s_μ of σ_μ.‡

(2) If $f(x, y)$ is absolutely continuous in x for every y and if $\partial f/\partial x$ is summable superficially, then, if y is given, for nearly every value of x,

$$\frac{\partial}{\partial x}f^{(\mu)} = \left(\frac{\partial f}{\partial x}\right)^{(\mu)}.$$

(3) If $f(x, y)$ is a potential function for its generalized derivatives, then, if y is given, for nearly every x

$$\frac{\partial}{\partial x}f^{(\mu)} = (D_x f)^\mu.$$

(4) If $f(x, y)$, besides satisfying the conditions of (3), is also

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§ G. C. Evans, *Fundamental problems of potential theory*, RICE INSTITUTE PAMPHLETS, vol. 7, No. 4 (Oct., 1920). In the original definition Professor Evans considers a general class Γ of curves. A summary of this paper is given in PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 7 (1921).

* Here, again, the definition as given by Professor Evans refers to the general class Γ of curves.

† The proofs are simple and are omitted, in most cases, for brevity.

‡ $f^{(\mu)}$ can be generalized when higher derivatives are required; e.g.,

$$f^\mu(x, y) = \frac{1}{\sigma_\mu{}^2}\int\int_{\sigma_\mu} d\xi' d\eta' \int\int_{\sigma_\mu} f(x + \xi + \xi', y + \eta + \eta')d\xi d\eta$$

possesses second derivatives nearly everywhere.

continuous in (x, y), then

$$\frac{\partial}{\partial x} f^{(\mu)} = (D_x f)^{(\mu)},$$

at every point (x, y).

(5) If $f(x, y)$ is a potential function for its generalized derivatives, then $D_x(f^{(\mu)})$ exists and is equal to $(D_x f)^{(\mu)}$.

(6) If $f(x, y)$ is summable superficially, the function $f(x+\xi, y+\eta)$ is measurable and summable in the three-dimensional region $[a \leqq s \leqq b, -\mu \leqq \xi \leqq \mu, -\mu \leqq \eta \leqq \mu]$, where s is the arc of a rectifiable curve upon which the point (x, y) lies, and

$$\int_{s_1}^{s_2} f^{(\mu)}(x, y)ds = \frac{1}{\sigma_\mu}\int_{s_1}^{s_2} ds \int\int_{\sigma_\mu} f(x+\xi, y+\eta)d\xi d\eta$$
$$= \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} d\xi d\eta \int_{s_1}^{s_2} f(x+\xi, y+\eta)ds.$$

(7) Similarly $f(x+\xi, y+\eta)$, regarded as a function of four variables (x, y, ξ, η), is summable in the region

$$[a \leqq x \leqq b, \quad c \leqq y \leqq d, \quad -\mu \leqq \xi \leqq \mu, \quad -\mu \leqq \eta \leqq \mu].$$

Consequently if σ is the rectangle $[a \leqq x \leqq b, \ c \leqq y \leqq d]$

$$\int\int_{\sigma} f^{(\mu)}(x, y)dxdy = \frac{1}{\sigma_\mu}\int\int_{\sigma} dxdy \int\int_{\sigma_\mu} f(x+\xi, y+\eta)d\xi d\eta$$
$$= \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} d\xi d\eta \int\int_{\sigma} f(x+\xi, y+\eta)dxdy.$$

(8) Since $\int\int_\sigma f(x+\xi, y+\eta)dxdy$ is a continuous function of (ξ, η),

$$\lim_{\mu\to 0}\int\int_{\sigma} f^{(\mu)}(x, y)dxdy$$
$$= \lim_{\sigma_\mu \to 0}\frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} d\xi d\eta \int\int_{\sigma} f(x+\xi, y+\eta)dxdy$$
$$= \int\int_{\sigma} f(x, y)dxdy.$$

(9) If $f(x, y)$ and $g(x, y)$ are two functions, summable with their squares, then

$$\lim_{\mu\to 0}\int\int_{\sigma} f^{(\mu)}g^{(\mu)}dxdy = \int\int_{\sigma} fgdxdy.$$

In fact since f^2 and g^2 are summable, it can be shown by

Schwarz's inequality that the absolute continuity of the integral $\int\int |f^{(\mu)}g^{(\mu)}|dxdy$ is uniform * for all μ, and this is sufficient to prove the property.

(10) If $f(x, y)$ is continuous in (x, y), $f^{(\mu)}$ approaches f uniformly as μ approaches 0. If f is merely summable, $f^{(\mu)}$ approaches f at nearly every point (x, y).

(11) If $f(x, y)$ is a function of limited variation in x, uniformly for every y, then $f^{(\mu)}$ is a function of limited variation in x, uniformly for all values of y and μ. In fact, if

$$\sum_i |f(x_{i+1}, y) - f(x_i, y)| \leqq T_x,$$

a constant, then

$$\sum_i |f^{(\mu)}(x_{i+1}, y) - f^{(\mu)}(x_i, y)|$$

$$\leqq \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu}\sum_i |f(x_{i+1}+\xi, y+\eta)-f(x_i+\xi, y+\eta)|d\xi d\eta \leqq T_x.\dagger$$

4. *Proof of the Formula.* It has been shown by W. H. Young ‡ that, given the continuous transformation $x = x(u, v)$, $y = y(u, v)$, the area of the image of the rectangle R $[a \leqq u \leqq b, c \leqq v \leqq d]$ is given by the formula

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)}\, dudv.$$

He assumes the absolute continuity of x and y as functions of u alone and of v alone, the absolute continuity with regard to u being uniform for a dense set of values of v in the interval $c \leqq v \leqq d$. In the theorem which follows absolute continuity of x and y is not assumed; instead it is assumed that x and y are potential functions for their generalized derivatives.

THEOREM. *If $x(u, v)$, $y(u, v)$ are continuous functions of (u, v), if each is a potential function for its generalized derivatives, the latter being summable with their squares, and if y (or x) is a function of uniformly limited variation in u for every v, and in v for every u, then the area of the image of the rectangle*

* C. de la Vallée-Poussin, TRANSACTIONS OF THIS SOCIETY, vol. 16 (1915), pp. 445 et seq.

† It is sufficient to assume that the total variation with regard to x is less than some function of y, $c(y)$, summable in y and such that

$$\int_y^{y'} c(y)dy < M(y' - y).$$

‡ W. H. Young, loc. cit.

$R\,[a \leqq u \leqq b,\ c \leqq v \leqq d]$ *is given by the formula*

$$A = \int\int_R (D_u x D_v y - D_v x D_u y)\,du\,dv.$$

It is shown by Young * that the area A is equal to

$$\tfrac{1}{2}\int_{C_R} x\,dy - y\,dx,$$

where the integrals are Stieltjes integrals evaluated about the contour C_R of R. By a known property of the Stieltjes integral,† namely,

$$\int_a^b x\,dy = xy\Big]_a^b - \int_a^b y\,dx,$$

it follows, under our hypothesis, that

$$A = \int_{C_R} x\,dy.$$

Instead of this expression we consider the approximation

$$A_\mu = \int_{C_R} x^{(\mu)} dy^{(\mu)} = \int_{C_R} x^{(\mu)} \frac{\partial}{\partial u} y^{(\mu)} du + x^{(\mu)} \frac{\partial}{\partial v} y^{(\mu)} dv.$$

The change to the last form is justified by the fact that $x^{(\mu)}$, $y^{(\mu)}$ are continuous with their first derivatives. By property (4) this can be written in the form

$$\int_{C_R} x^{(\mu)} (D_u y)^{(\mu)} du + x^{(\mu)} (D_v y)^{(\mu)} dv,$$

and since the integrands are absolutely continuous

$$A_\mu = \int_a^b du \int_c^d -\frac{\partial}{\partial v}[x^{(\mu)}(D_u y)^{(\mu)}]dv + \int_c^d dv \int_a^b \frac{\partial}{\partial u}[x^{(\mu)}(D_v y)^{(\mu)}]du.$$

The quantity $\partial(D_v y)^{(\mu)}/\partial u$ is summable superficially, being equal to the contour integral of $D_v y$ about a square (properties (1) and (6)). Moreover for every rectangle $r\,[a' \leqq u \leqq b',\ c' \leqq v \leqq d']$,

$$\int\int_r \left[\frac{\partial}{\partial u}(D_v y)^{(\mu)}\right] du\,dv = \int\int_r \left[\frac{\partial}{\partial v}(D_u y)^{(\mu)}\right] du\,dv,$$

* W. H. Young, loc. cit.

† H. E. Bray, Annals of Mathematics, (2), vol. 20 (1918-19), p. 185.

Since

$$\int\int_r \left[\frac{\partial}{\partial u}(D_v y)^{(\mu)}\right] dudv = \left[\int_{c'}^{d'} (D_v y)^\mu dv\right]_{u=a'}^{u=b'}$$
$$= \left[\int_{c'}^{d'} \frac{\partial}{\partial v} y^{(\mu)} dv\right]_{u=a'}^{u=b'}$$
$$= y^{(\mu)}(b', d') - y^{(\mu)}(b', c') - y^{(\mu)}(a', d') + y^{(\mu)}(a', c').$$

Consequently, x being continuous, for every rectangle r,

$$\int\int_r x^{(\mu)} \frac{\partial}{\partial u}(D_v y)^{(\mu)} dudv = \int\int_r x^{(\mu)} \frac{\partial}{\partial v}(D_u y)^{(\mu)} dudv.$$

We can now express the quantity A_μ as a double integral and on cancelling the terms just mentioned we obtain

$$A_\mu = \int\int_R \left[\frac{\partial}{\partial u} x^{(\mu)} (D_v y)^{(\mu)} - \frac{\partial}{\partial v} x^{(\mu)} (D_u y)^{(\mu)}\right] dudv,$$

and since

$$\frac{\partial}{\partial u} x^{(\mu)} = (D_u x)^{(\mu)}, \qquad \frac{\partial}{\partial v} x^{(\mu)} = (D_v x)^{(\mu)},$$

by property (4),

$$A_\mu = \int\int_R [(D_u x)^{(\mu)}(D_v y)^{(\mu)} - (D_v x)^{(\mu)}(D_u y)^{(\mu)}] dudv.$$

Now let μ approach zero. Since the generalized derivatives are, by hypothesis, summable with their squares, by (9),

$$\lim_{\mu \to 0} A_\mu = \int\int_R [D_u x D_v y - D_v x D_u y] dudv.$$

To show that $\lim_{\mu\to 0} A_\mu = \int_{C_R} xdy$, consider the following quantity, in which the integrals are taken in regard to u for a fixed value of v:

$$\left|\int_{u=a}^{u=b} x^{(\mu)} dy^{(\mu)} - \int_{u=a}^{u=b} xdy\right|$$
$$\leqq \left|\int x^{(\mu)} dy^{(\mu)} - \int xdy^{(\mu)}\right| + \left|\int xdy^{(\mu)} - \int xdy\right|$$
$$= \left|\int (x^\mu - x) dy^{(\mu)}\right| + \left|\int xdy^{(\mu)} - \int xdy\right| \cdot$$

The first term is not greater than $\max|x^{(\mu)} - x|$ times the total variation of $y^{(\mu)}$; and since the latter is uniformly limited for every μ and $x^{(\mu)}$ approaches x uniformly, this term approaches zero with μ. The second term approaches zero because $y^{(\mu)}$ approaches y and is of uniformly limited variation in u for every μ,* by (11). The quantity $\int_{u=a}^{u=b} x^{(\mu)}dy^{(\mu)}$ is typical of those which constitute A_μ, consequently

$$\lim_{\mu \to 0} A_\mu = \int_{C_R} xdy = A.$$

We have thus proved that

$$A = \int\int [D_u x D_v y - D_v x D_u y] dudv.$$

It has been shown by Evans † that if a continuous function is a potential function for its generalized derivatives, then its ordinary derivatives exist nearly everywhere and are equal to the corresponding generalized derivatives nearly everywhere. The formula which we have proved remains true, consequently, if ordinary derivatives are substituted for the generalized derivatives. On the other hand, if $\partial x/\partial u$, $\partial x/\partial v$ are given, a vector φ is defined whose components in the u and v directions are $\partial x/\partial u$, $\partial x/\partial v$, respectively, and

$$\varphi_\alpha = \frac{\partial x}{\partial u} \cos (u, \alpha) + \frac{\partial x}{\partial v} \sin (u, \alpha).$$

If $\partial x/\partial u$, $\partial x/\partial v$ are summable the same is true for any component φ_α and if

$$\int\int_\sigma \frac{\partial x}{\partial u} dudv = \int_{C_\sigma} xdv, \qquad \int\int_\sigma \frac{\partial x}{\partial v} dudv = -\int_{C_\sigma} xdu,$$

x will be a potential function for the vector φ which is called its gradient vector.‡ Hence we have the following theorem.

THEOREM. *If $x(u, v)$, $y(u, v)$ are continuous functions of (u, v), if each is a potential function for its gradient vector, if y (or x) is of uniformly limited variation in u for every v and in v for every u, and if $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, $\partial y/\partial v$ are summable with their squares, then*

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)} dudv.$$

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* H. E. Bray, loc. cit.

† G. C. Evans, RICE INSTITUTE PAMPHLETS, vol. 7, No. 4, pp. 97–99.

‡ G. C. Evans, loc. cit.