NOTE ON THE CONVERGENCE OF WEIGHTED TRIGONOMETRIC SERIES*

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1. *Introduction.* Let $f(x)$ be a function continuous for all values of x, and of period 2π. Let $T_n(x)$ be a trigonometric sum of the nth order.† If $T_n(x)$ is determined, among all such sums, by the condition that the value of the integral

$$\int_0^{2\pi} [f(x) - T_n(x)]^2 dx$$

shall be a minimum, it becomes the partial sum of the Fourier series for $f(x)$. The problem can be generalized by taking, as the quantity to be reduced to a minimum, the integral

$$\int_0^{2\pi} \rho(x)[f(x) - T_n(x)]^2 dx, \tag{1}$$

where $\rho(x)$, indicating the *weight* to be attached to different values of the argument, is a function of x, likewise of period 2π, and positive for all values of x. There is a considerable body of literature bearing more or less directly on the generalized problem. This literature owes its inspiration largely to the researches of Tchebychef;‡ particular mention should also be made of a classical memoir by Gram.§

The purpose of the following paragraphs is to discuss the convergence of $T_n(x)$ toward the value $f(x)$, as n becomes infinite. The method is one which I have used recently in connection with the corresponding problem in which the weight is constantly equal to unity, and the square of the error is replaced by a power with a different exponent. The

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† The words "of the nth order" will be understood throughout to mean "of the nth order at most."

‡ Cf., e.g., H. Burkhardt, *Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik*, JAHRESBERICHT DER VEREINIGUNG, vol. 10, Heft 2 (1908), pp. 823 ff.

§ J. P. Gram, *Ueber die Entwickelung reeller Functionen in Reihen mittelst der Methode der kleinsten Quadrate*, JOURNAL FÜR MATHEMATIK, vol. 94 (1883), pp. 41–73.

question of convergence is treated by Gram, in the paper cited, but scarcely in a manner to meet the requirements of modern analysis.* More recently it has come within the range of a number of investigations, including a series of papers by Stekloff, in the BULLETIN DE L'ACADÉMIE DES SCIENCES, PETROGRAD, and elsewhere, with which I am only very imperfectly acquainted; a paper by J. Chokhate,† which I have seen in manuscript; and a series of papers by Szegö.‡ Up to the present time, I have not seen any treatment covering precisely the results that are presented below. If it should appear nevertheless that such a treatment exists, the novelty of this paper would consist in the method employed, and in the applicability of the method to the case in which the exponent 2 in (1) is replaced by an arbitrary m, as suggested in the concluding paragraph.

2. *The Convergence Theorem.* The conclusion to be established is as follows: §

Let $\omega(\delta)$ be the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. Let $\rho(x)$ be continuous and positive for all values of x; or, if not continuous, let it be measurable, and always included between two fixed positive bounds.|| Then we may state the theorem:

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* Cf. Burkhardt, loc. cit., pp. 848–854.

† See also J. Chokhate, *Sur quelques propriétés des polynomes de Tchébicheff*, COMPTES RENDUS, vol. 166 (1918), pp. 28–31.

‡ G. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, MATHEMATISCHE ANNALEN, vol. 82 (1921), pp. 188–212; *Über die Entwicklung einer willkürlichen Funktion nach den Polynomen eines Orthogonalsystems*, MATHEMATISCHE ZEITSCHRIFT, vol. 12 (1922), pp. 61–94; *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, MATHEMATISCHE ANNALEN, vol. 86 (1922), pp. 114–139; and other papers referred to in footnotes attached to the above.

§ The proof of the existence of a unique solution for the minimum problem is based so directly on similar proofs already given that it will not be taken up in detail here; cf. D. Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117–128.

|| It would of course make no difference if this condition were violated at points of a set of measure zero, since the value of the integral (1), and consequently the determination of $T_n(x)$, would not be affected.

The sum $T_n(x)$ *will converge uniformly to the value* $f(x)$ *for* $n \to \infty$ *provided that**

$$\lim_{\delta \to 0} \omega(\delta)/\sqrt{\delta} = 0.$$

As already stated, the proof is similar to one given recently in another connection.† In the first place, if $f(x)$ and $\varphi(x)$ are two functions whose difference is a trigonometric sum $t_n(x)$ of order n:

$$f(x) = \varphi(x) + t_n(x),$$

and if $T_n(x)$ and $\tau_n(x)$ are two sums, likewise of order n, such that

$$T_n(x) = \tau_n(x) + t_n(x),$$

the value of the integral (1) formed with $f(x)$ and $T_n(x)$ is the same as the value of the corresponding integral formed with $\varphi(x)$ and $\tau_n(x)$, and both integrals will reach their minimum values simultaneously. That is, if $T_n(x)$ and $\tau_n(x)$ represent the best approximating functions for $f(x)$ and $\varphi(x)$, respectively, as judged by the value of the integral (1), the errors $f(x) - T_n(x)$ and $\varphi(x) - \tau_n(x)$ will be identical.

By a general theorem on the approximate representation of continuous functions,‡ there will exist sums $t_n(x)$, of all orders $n > 0$, such that the difference between $f(x)$ and $t_n(x)$ never exceeds a constant multiple of $\omega(2\pi/n)$. In formulas, let

$$\varphi_n(x) = f(x) - t_n(x),$$

and let ϵ_n be the maximum of $|\varphi_n(x)|$; then

$$\epsilon_n \leqq c\omega(2\pi/n),$$

where c is independent of n. In particular, if $\omega(\delta)$ satisfies

* There is no reason to suppose that the particular infinitesimal $\sqrt{\delta}$ has any essential significance for the problem; its occurrence is in all probability due merely to the limitations of the method.

† D. Jackson, *On the convergence of certain trigonometric and polynomial approximations*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 158–166.

‡ Cf., e.g., D. Jackson, *On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 343–364; p. 350.

the hypothesis of the theorem,

$$\lim_{n\to\infty} \epsilon \sqrt{n} = 0. \tag{2}$$

Let $\tau_n(x)$ be the trigonometric sum of order n which gives the best approximation to $\varphi_n(x)$, as determined by the integral corresponding to (1); let

$$\gamma_n = \int_0^{2\pi} \rho(x)[\varphi_n(x) - \tau_n(x)]^2 dx;$$

and let $\mu_n = |\tau_n(x_0)|$ be the maximum of $|\tau_n(x)|$. Let it be assumed that

$$0 < v \leqq \rho(x) \leqq V,$$

the numbers v and V being constants; and let it be assumed temporarily that $\mu_n \geqq 4\epsilon_n$.

By Bernstein's theorem,* since

$$|\tau_n(x)| \leqq \mu_n,$$

it follows that

$$|\tau_n'(x)| \leqq n\mu_n$$

for all values of x. In particular, for values of x in the interval

$$|x - x_0| \leqq \frac{1}{2n},$$

it can be inferred that

$$|\tau_n(x) - \tau_n(x_0)| \leqq \frac{\mu_n}{2},$$

and

$$|\tau_n(x)| \geqq \frac{\mu_n}{2}.$$

Since

$$|\varphi_n(x)| \leqq \epsilon_n \leqq \mu_n/4,$$

it follows further that

$$|\varphi_n(x) - \tau_n(x)| \geqq \frac{\mu_n}{4}$$

throughout the interval specified, and, as the length of the interval is $1/n$, and $\rho(x) \geqq v$,

$$\gamma_n \geqq \frac{v}{n}\left(\frac{\mu_n}{4}\right)^2.$$

* See, e.g., de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, Paris, 1919, pp. 39–42.

On the other hand, by the minimum property of $\tau_n(x)$, the value of γ_n is less than that which would be obtained if $\tau_n(x)$ were replaced by any other trigonometric sum of order n; in particular, by comparison with the integral which is obtained if 0 is substituted for $\tau_n(x)$,

$$\gamma_n \leqq 2\pi V \epsilon_n^2.$$

Hence

$$\frac{v}{n}\left(\frac{\mu_n}{4}\right)^2 \leqq 2\pi V \epsilon_n^2,$$

$$\mu_n \leqq 4\sqrt{\frac{2\pi V}{v}}\ \epsilon_n \sqrt{n}.$$

This relation, derived on the hypothesis that $\mu_n \geqq 4\epsilon_n$, clearly holds in the contrary case also, since $V \geqq v$ and $n \geqq 1$.

In any case, then, since $|\varphi_n| \leqq \epsilon_n$ and $|\tau_n| \leqq \mu_n$,

$$|\varphi_n(x) - \tau_n(x)| \leqq \epsilon_n + 4\sqrt{\frac{2\pi V}{v}}\ \epsilon_n \sqrt{n} \leqq k\epsilon_n \sqrt{n},$$

where k is independent of n. But it has been pointed out already that $\varphi_n(x) - \tau_n(x)$ is the same as $f(x) - T_n(x)$, where $T_n(x)$ is the sum giving the best approximation to $f(x)$, as determined by the integral (1); hence

$$|f(x) - T_n(x)| \leqq k\epsilon_n \sqrt{n}.$$

This relation, combined with (2), establishes the truth of the theorem.

With the same method of treatment, the problem can be varied by using a general power of the absolute value of the error, instead of the square, together with a weight-function $\rho(x)$; and the method is applicable also to problems of polynomial approximation. For treatment in detail, however, the case discussed above may be regarded as sufficiently illustrative.

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