

THE APRIL MEETING OF THE SOCIETY IN NEW YORK

The two hundred twenty-ninth regular meeting of the Society was held at Columbia University, on Saturday, April 28, 1923, extending through the usual morning and afternoon sessions. The attendance included the following fifty-seven members:

Archibald, Beal, Birkhoff, Blichfeldt, B. H. Camp, Cole, Crum, Dantzig, Dodd, Douglas, Fine, Fiske, Fite, M. C. Foster, Philip Franklin, Fry, Gafafer, Gill, Gronwall, Hazlett, Hebbert, E. R. Hedrick, Hille, Huntington, Joffe, Kellogg, Kircher, Kuhn, Lamson, MacDuffee, Mathewson, Meder, Mirick, Molina, Mullins, Northcott, Oglesby, Osgood, Pell, Pfeiffer, Raynor, Reddick, R. G. D. Richardson, Ritt, Rosenbaum, Seely, Sicheloff, Silverman, Sosnow, Swartzel, Tracey, Vandiver, H. E. Webb, Weisner, Wetzel, H. S. White, J. W. Young.

At the meeting of the Council, the following twenty persons were elected to membership in the Society:

Professor Thomas Cicero Amick, Elon College;
 Professor Harry Cyrus Bradley, Massachusetts Institute of Technology;
 Mr. Alexander Joseph Cook, Harvard University;
 Professor Palmer Hampton Graham, New York University;
 Professor David Arthur Hatch, Lafayette College;
 Mr. Harold Hotelling, Princeton University;
 Professor Emma Hyde, Kansas State Agricultural College;
 Mr. Philip Chapin Jones, Goodyear Tire and Rubber Company;
 Mr. Ralph Eugene Kennon, University of Iowa;
 Mr. Edgar Lucien Larkin, Lowe Astronomical Observatory;
 Mr. Ben Zion Linfield, Harvard University;
 Mr. George Averett Lyle, Lehigh University;
 Mr. Dalip Singh Saund, University of California;
 Mr. David Skolnik, Central High School, Newark;
 Mr. Han Yih Tsang, Columbia University;
 Mr. John Isaac Vass, Northwestern University;
 Professor Evelyn Walker, Hunter College;
 Mr. Albert Harry Wheeler, North High School, Worcester;
 Professor Hugh Brown Wilcox, University of Minnesota;
 Mr. Kitizi Yanagihara, Yamagata Higher School.

The Secretary announced that the following members of the London Mathematical Society had entered the American Mathematical Society since the February meeting, under the reciprocity agreement:

Dr. Bevan Braithwaite Baker, University of Edinburgh;

Professor John Edward Aloysius Steggall, University College, Dundee;
Professor William Henry Young, University College, Aberystwyth.

Twenty-seven applications for membership in the Society were received.

The Secretary reported the appointment by President Veblen of the following Committee on Endowment: Professors J. L. Coolidge (chairman), Arnold Dresden, and G. C. Evans, and Messrs. Robert Henderson and G. E. Roosevelt (treasurer). The Council approved a statement of uses to which income from the proposed endowment may be devoted. Briefly these include the subsidizing of the *TRANSACTIONS*, the Colloquium Lectures, and other research publications, together with treatises on advanced topics. A statement from Professor Coolidge outlining the proposed programme of the Committee was read.

Professor Edward Kasner was elected a member of the Editorial Committee of the *TRANSACTIONS*, as successor to Professor L. P. Eisenhart, who had declined renomination.

Professor Eisenhart was elected to complete the term of service of Professor C. N. Haskins as representative of the Society in the Division of Physical Sciences of the National Research Council, Professor Haskins having asked to be relieved because of ill health.

The Secretary reported that the Cole Prize Fund (see this *BULLETIN*, vol. 29, p. 14) had been augmented to more than one thousand dollars. The following committee was appointed to set the first prize problem and to arrange the conditions of award: Professors H. S. White (chairman), Blichfeldt, Dickson, Fiske, and Osgood.

About fifty members took luncheon together between the sessions, and fifteen gathered at the dinner after the meeting.

Professor H. S. White presided at the morning session, relieved in the afternoon by Professor E. V. Huntington. Titles and abstracts of the papers read at this meeting follow below. Mr. Michal was introduced by Professor Evans, Professor Nörlund by Professor Birkhoff, Dr. Rainich by Professor Kasner, and Mr. Levy by Professor Eisenhart.

Professor Haskins' paper was read by Professor Silverman, and the papers of Bliss, Hutchinson, Glenn, Gronwall (first paper), Douglas (second paper), Evans and Bray, Michal, Wiener, Walsh, Garabedian, Safford, Graustein, Nörlund, Eisenhart, Brinkmann, Lipka and Hollcroft, Zeldin, and Veblen and Thomas were read by title.

1. Professor W. L. Crum: *The resemblance between the ordinate of the periodogram and the correlation coefficient.*

This paper points out that the ordinate of the periodogram, as now defined, does not differ materially from the product-sum which occurs in the numerator of the correlation coefficient for the original data with a periodic function of the simplest trigonometric type. This fact indicates that a slight change in the definition of the periodogram may be desirable, and suggests also an ideal means of studying the lag in the correlation of two historical series.

2. Professor G. A. Bliss: *Birational transformations simplifying singularities of algebraic curves.*

There is a well known theorem in the theory of algebraic curves which states that every such curve can be transformed by a birational transformation into one which has no singularities except double points with distinct tangents. The theorem is not a simple one to demonstrate, and many of the proofs which have been given are incomplete or inaccurate. In a preceding paper* the author has commented upon these proofs and has signalized two of them as being especially interesting. One is by Walker, who developed an alteration, suggested by Klein in 1894, of a method originally devised by Bertini for the projective plane. In the second, by Hensel and Landsberg, reasoning proposed by Kronecker in 1881 is extended to apply to curves in the function-theoretic plane. Both of these proofs are lengthy and complicated when all the details are taken into consideration. In the present paper the author has remodeled the method of Kronecker so that it can be applied to both planes, and has attained what he hopes will be regarded as simpler proofs of the two corresponding theorems.

* *The reduction of singularities of plane curves by birational transformations*, this BULLETIN, vol. 29 (1923), pp. 161-183.

3. Professor O. D. Kellogg: *Curvature and the top.*

If a plane or spherical curve has curvature, K , which has a continuous derivative with respect to the arc, s , and if for $a \leq s \leq b$ this derivative is positive, the osculating circle at the point s_2 lies within the osculating circle at s_1 if $a \leq s_1 < s_2 \leq b$. The present paper proves this theorem, and gives applications to the theory of the top suggested by Professor Osgood's article in the April number of the TRANSACTIONS OF THIS SOCIETY (vol. 23, No. 3).

4. Professor E. V. Huntington: *Simplified proof of l'Hospital's theorem on indeterminate forms.*

L'Hospital's familiar theorem on indeterminate forms is usually based on the generalized law of the mean, the proof of which often appears to the student as a highly ingenious device. The present author's proof of this theorem depends only on the most elementary properties of integrals, and the motive for each step taken is comparatively obvious. The theorem is stated in an extended form so as to include curves with vertical points of inflection, and new examples are given to show the necessity of the condition that the derivative of the denominator must not vanish in the neighborhood of the point in question.

5. Professor E. V. Huntington: *Tables of Lagrangean coefficients, for interpolating without differences.*

The chief advantages of the Lagrangean formula for interpolation (for equal intervals) are (1) that the required interpolated value is expressed directly in terms of the tabulated values, without the necessity of forming a table of differences, and (2) that the formula, unlike those of Stirling, Bessel, etc., can be written down at once, without effort, and verified by inspection. The chief disadvantages are (1) the fact that the numerical factors required are not small numbers as in the case of the ordinary formulas, and (2) that tables of the coefficients have not hitherto been available. The first of these disadvantages has largely disappeared with the advent of computing machines, and the second is removed by the tables presented by the present author. By the aid of these tables an interpolated value can be computed on the machine in less time than it takes to write out the differences. The tables are arranged for polynomials of the third, fourth, and fifth degrees.

6. Professor J. I. Hutchinson: *A remarkable class of entire functions.*

If $f(z) = a_0 + a_1z + \dots + a_nz^n + \dots$ has real and positive coefficients and if $a_n^{-1} = b_1b_2 \dots b_n$, it is proved that $b_n \cong 4b_{n-1}$ ($n = 2, 3, \dots$) are necessary and sufficient conditions that $f(z)$, as well as any polynomial formed by any number of consecutive terms of $f(z)$, has all its roots real and distinct (except $z = 0$, when it is a multiple root). Various properties and special cases are considered.

7. Professor C. N. Haskins: *On Newton's formulas for the sums of the powers of the roots of an algebraic equation.*

The theorem of this note was brought out by an attempt to simplify and check Bourguet's* computation of the coefficients of the power-series development of the Γ function.

8. Professor O. E. Glenn: *Invariants of the transformation of a differential form by analytic transformations.* Preliminary report.

It is known that restrictions which reduce the number of operations of the group or set of transformations applied to a quantic enlarge the totality of independent concomitants. If conditions are placed upon the arbitrary functions in the transformations for a binary differential form, such as to make these functions satisfy the pair of conditions for being analytic, the differential parameters increase in numbers and new methods are required. The author develops theory for the identification of complete systems in certain domains.

9. Dr. T. H. Gronwall: *Mutual induction of two square coils.*

This paper gives a method for the rapid numerical computation of the mutual induction of two square coils with parallel axes, one coil being rotated through any angle about its axis. Such an arrangement is used in a certain type of radio antenna. The paper will appear in the BELL SYSTEM TECHNICAL JOURNAL.

10. Dr. Einar Hille: *On Dirichlet's series with complex exponents.*

The author considers series of the type $\sum a_n e^{-\lambda_n s}$, where the exponents λ_n form a sequence of complex numbers such that $|\lambda_n| \rightarrow \infty$. It is shown that the region of absolute convergence of such a series is a convex domain. This region

* ANNALES DE L'ECOLE NORMALE, (2), vol. 10 (1881), pp. 175-233.

is determined in terms of the a_n and the λ_n under different assumptions concerning the exponents. For every convex region Dirichlet's series can be constructed, having the region as region of absolute convergence. The results are extended to Dirichlet's integrals.

11. Dr. Jesse Douglas: *Normal congruences and quadruply infinite families of curves in space*. Second paper.

In a previous paper of the same title (this BULLETIN, vol. 28, p. 238) the author classified quadruply infinite families of space curves with respect to the normal congruences contained within them. In that classification, two types of curve family, termed natural and quasi-natural, were predominant. The present paper solves the following problem: To determine, for each of the following four properties, all families of ∞^4 curves in space which have that property. (1) The ∞^2 curves of the family which pass through an arbitrary point form a normal congruence. (2) The ∞^2 curves of the family which meet an arbitrary plane perpendicularly form a normal congruence. (3) The ∞^2 curves of the family which meet an arbitrary sphere perpendicularly form a normal congruence. (4) The ∞^2 curves of the family which meet an arbitrary straight line perpendicularly form a normal congruence. There are three types with the property (1), of which two are the natural and quasi-natural. Five types, including the natural and quasi-natural, have the property (2). All and only natural and quasi-natural families have the property (3). All and only natural families have the property (4).

12. Dr. Jesse Douglas: *Determination of all families of ∞^4 curves in space in which the sum of the angles of every triangle is two right angles*.

This paper will appear in full in an early issue of this BULLETIN.

13. Dr. C. C. MacDuffee: *On the complete independence of the functional equations of involution*.

The functional equations (1) $\varphi(a, m) \cdot \varphi(a, n) = \varphi(a, m + n)$, (2) $\varphi(\varphi(a, m), n) = \varphi(a, mn)$, (3) $\varphi(a, n) \cdot \varphi(b, n) = \varphi(ab, n)$ are satisfied unreservedly by the numbers of the class C_+ of positive numbers and zero, when $\varphi(a, n)$ is identified with the power function a^n . The complex numbers satisfy these equations when proper determinations of the multiple-valued power functions are used. In this paper, a non-linear algebra

N is introduced whose elements are the pairs of real numbers $[\xi, \tau]$. Two operations, addition and multiplication, are defined, and it is shown that these operations obey all the ordinary laws of algebra except the associative law of addition. This algebra N is then used in the discussion of the complete independence of equations (1), (2) and (3). With certain underlying conditions, it is proved that the functional equations are not completely independent, for (1) and (2) imply (3). The remaining seven cases are non-empty, for seven functions are defined on the algebra N which obey the underlying conditions, each fulfilling one of these seven cases.

14. Professor G. C. Evans and Dr. H. E. Bray: *The Dirichlet problem for the sphere, and its generalization; necessary and sufficient conditions.*

The authors find necessary and sufficient conditions that a function harmonic within a sphere, not necessarily bounded, may be written as a Poisson integral in terms of boundary values f summable on the surface of the sphere in the Lebesgue sense; also, more generally, as a Poisson integral in terms of the differential of an additive function of point sets $F(\omega)$ on the surface of the sphere. Extending the result of Fatou, they show that the harmonic function takes on the value of the point set derivative of $F(\omega)$ wherever the latter exists, and hence, in the former problem, the value of f almost everywhere. A physical interpretation in terms of sources and doublets is given for the extended problem, using a new theorem of Vitali on the analysis of continuous functions of limited variation with zero derivative almost everywhere.

15. Mr. A. Michal: *Integro-differential invariants of one-parameter groups of Volterra transformations.* Preliminary report.

The author commences the study of a problem allied to the calculus of variations, by showing that there always exist functionals of a function y and its derivative which are invariant under a given arbitrary continuous one-parameter group of linear functional transformations of Volterra type of the argument y . The functional is assumed to involve y and its derivative between the values 0 and x , and, considered as a functional of these two arguments, may for a given functional value of the derivative of y be assigned arbitrarily as a functional of the other argument.

16. Dr. Norbert Wiener: *A generalization of the Dirichlet problem.*

The author uses the methods of generalized integration due to Daniell to extend the notion of a harmonic function corresponding to given boundary conditions to discontinuous boundary conditions of a very general type. Existence theorems are given.

17. Dr. Norbert Wiener: *A new type of summability.*

The author develops a type of summability with trigonometric weighting factors, and shows that it correctly evaluates the Fourier series of a continuous function. This type of summability is not stronger than the first type of summability of Cesàro.

18. Dr. J. L. Walsh: *On the location of the roots of Lamé's polynomials.*

A certain problem in the statical equilibrium of particles arises naturally in connection with the theory of equations. This same problem, in a generalized form, arises also in connection with the location of the roots of certain polynomial solutions of a differential equation considered by Lamé. This paper extends to the latter type of problem some simple results recently established for the former type.

19. Mr. C. A. Garabedian: *Rods of constant or variable circular cross section.*

In a paper on *Circular plates of constant or variable thickness*, presented to the Society February 24, 1923, the author developed a method of series in elasticity, and called attention to the possibility of treating by the same method the problem of the thin rod. Cylindrical coordinates are adopted, and the rod is generated by revolution of a curve $r = f(z)$ about the axis of z . The chief interest lies in the fact that the section may vary with z . Writing $r = \rho\tau$, $f(z) = \varphi(z)\tau$, it is assumed that the displacements are developable in ascending powers of the parameter τ . The equations of equilibrium become identities on τ , and computation of the coefficients in the formulas of displacement is direct. In prescribing boundary conditions at the ends, conventional use is made of Saint-Venant's principle. The total differential equations which arise can be integrated explicitly in a wide range of cases. Applications are made to the uniformly tapering rod, to a bulging rod, and to the right circular cylinder.

20. Professor F. H. Safford: *A pendulum of varying length.*

This paper was written in answer to a request to obtain the law of variation in length if the amplitude of swing is to increase as rapidly as possible. In solving the differential equation of motion, it was found that a change of the independent variable time could be made which reduced the problem to that of a simple pendulum of fixed length. The amplitude may be made any assigned function of the time, and a criterion was obtained concerning the transition from vibration to complete revolution.

21. Professor W. C. Graustein: *Note on a certain type of ruled surface.*

This paper will appear in full in an early issue of this BULLETIN.

22. Professor N. E. Nörlund: *On certain difference equations.*

The object of this paper is to study the solutions of the difference equation $\Delta^n \omega F(x) = \varphi(x)$, where $\varphi(x)$ is a given function. By a certain method of summation the author defines an operation which is inverse to the difference operation Δ . He finds thus a particular solution of the above equation. This solution may also be defined by boundary value conditions. The paper contains a detailed discussion of the properties of this particular solution. The general solution differs from the above particular solution by an arbitrary function whose n th difference is zero.

23. Professor L. P. Eisenhart: *Orthogonal systems of hypersurfaces in a Riemann space.*

Let $g_{rs}dx^r dx^s$ be the fundamental quadratic form of a Riemann space of order n , and α_{rs} the covariant components of any symmetric tensor of the second order other than g_{rs} . The equations $(\alpha_{rs} + \rho g_{rs})\lambda^r = 0$ determine an n -uple of congruences whose directions at any point are mutually orthogonal, the functions $\lambda_h{}^r$ being the contravariant components of the tangents to a curve of the congruence C_h at any point. The necessary and sufficient conditions upon the tensors g_{rs} and α_{rs} in order that the congruences be normal, and consequently the space admit an n -uply orthogonal system of hypersurfaces, are determined in this paper. It is shown that the problem is an algebraic one. The conditions are simple when the roots of the equation $(\alpha_{rs} + \rho g_{rs}) = 0$ are simple or double at most, but are quite involved where there are roots of the third and higher orders.

24. Professor L. P. Eisenhart: *Symmetric tensors of the second order whose first covariant derivatives are zero.*

The following theorem is established: A necessary and sufficient condition that a Riemann space admit a symmetric covariant tensor of the second order α_{rs} other than the fundamental tensor of the space, g_{rs} , such that its first covariant derivative is zero, is that the fundamental quadratic form $g_{rs}dx^r dx^s$ be reducible to a sum of quadratic forms φ^i , the coefficients of each form being functions at most of the x 's of that form; then $\alpha_{rs}dx^r dx^s = \sum_i \rho_i \varphi^i$, where the ρ 's are arbitrary constants. The determination of whether or not a given space is of this kind is reducible to a problem of algebra.

25. Dr. G. Y. Rainich: *Geometry of curved space without coordinates.*

An n -dimensional bundle of vectors is introduced axiomatically, the properties of addition of vectors, multiplication of vectors with scalars, and formation of ratio serving as axioms. Space is considered as a totality of points; an axiom states that vectors issuing from one point constitute an n -dimensional bundle. This permits introducing a relation between different bundles. The transitivity of this relation is the condition that space be euclidean. In the general case the geometry in the vicinity of every point can be given by a trilinear vector function which is the generalized Riemann tensor. Scalar and vector multilinear functions are introduced as two kinds of tensors which are essentially distinct unless a length unit is introduced. Weyl's "Dichten" are shown on the contrary not to be essentially different from tensors. The notions of transformations, invariants, covariant and contravariant quantities are not essential to geometry and arise only with the introduction of coordinates. The metric does not define the relation between bundles, but for the representation of a metric we might use these relations, and in this case the Riemann tensor can be deduced from this representation.

26. Mr. Harry Levy: *Normal congruences of curves in a Riemann space.*

The following theorems are demonstrated: (1) A necessary and sufficient condition that a congruence of curves in a Riemann space of order n admit a family of r -dimensional ($r > 1$) hypersurfaces as orthogonal trajectories is that

$\gamma_{ijk} = \gamma_{ikj}$ ($i = 1, 2, \dots, n - r; j, k = n - r + 1, \dots, n$), where the functions γ_{ijk} have the significance given them by Ricci and Levi-Civita (MATHEMATISCHE ANNALEN, vol. 54 (1901), p. 148). (2) If each of n mutually orthogonal congruences has a family of r -dimensional hypersurfaces as orthogonal trajectories, and if the n families of hypersurfaces are distinct, the congruences are normal, that is, each has a family of $(n - 1)$ -dimensional hypersurfaces as orthogonal trajectories.

27. Dr. Philip Franklin: *Linear tensor equations.*

In this paper we investigate under what conditions a set of equations involving tensor components, invariant, as a set, under changes of coordinates, is equivalent to a set of tensor equations, i.e., equations expressing the vanishing of *all* the components of a tensor. We show that a set of equations *linear* in the components of a single tensor, with numerical coefficients, holding for all coordinates, is equivalent to a set of tensor equations. This theorem, proved for n -space, is applied to the proof that there are essentially only three distinct linear tensor equations in the curvature tensor, a theorem proved for 4-space by Birkhoff. A general theorem restricting the possible forms of linear tensor equations is added, which may be used to deduce such equations as those of the electromagnetic field.

28. Dr. Philip Franklin: *A qualitative definition of the potential functions.*

The purpose of this paper is to set up assumptions characterizing the potential functions which do not involve derivatives or integrals, and express obvious properties of the physical quantities giving rise to potential functions. It is shown that a class of functions of two variables such that its members are continuous, any linear combination of two members is in the class, any member gives rise to a new member if the coordinate axes are shifted (orthogonal transformation of the variables), it contains a constant function, and for an infinite sequence of functions of the class, taking values on the boundary of a fixed circle approaching zero uniformly over the boundary, the values at the center of the circle cannot approach a limit different from zero, *is necessarily a set of harmonic functions.* An analogous set of assumptions is given for potential functions of three variables.

29. Mr. H. W. Brinkmann: *Riemann spaces conformal to Einstein spaces.*

A Riemann n -space which satisfies those Einstein equations often referred to as the "cosmological equations" is called an *Einstein space*. The author obtains a necessary and sufficient condition that a given Riemann space be conformal to such a manifold. This is then applied to the special case where the given manifold is an Einstein space to start with. For the dimensionality $n = 4$ there is proved, among other things, the following theorem: If two four-dimensional Einstein spaces are mapped conformally they are either both of constant Riemann curvature (spherical spaces) or they are isometric and the conformal map in question is merely a change of scale. A note on this subject has been sent to the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES.

30. Mr. H. W. Brinkmann: *Einstein spaces mapped conformally on each other.*

In this paper the author determines all n -dimensional Einstein spaces which can be mapped conformally on another Einstein space provided that map which consists in a mere change of scale is ignored. The results are quite simple, and agree for $n = 4$ with the theorem stated in the preceding paper. It must be noted that the two Einstein manifolds, in question need not have the same scalar curvature, but if their scalar curvatures are both zero or both not zero it is shown that one of the two conformal spaces is isometric to that obtained from the other by a suitable change of scale. The isometric map is, however, not established by the conformal one. When one of the scalar curvatures vanishes and the other does not, it is evident that the two spaces can not be isometric even if a change of scale is allowed.

31. Dr. M. C. Foster: *Surfaces with orthogonal loci of the centers of geodesic curvature of an orthogonal system.*

The displacements of the centers of geodesic curvature are considered relative to the moving trihedral. The loci of these points are not surfaces but orthogonal curves of which at least one must be a straight line. The given orthogonal system must be the lines of curvature. Such surfaces are *surfaces of Joachimsthal* with circular lines of curvature in one system, and for which the fundamental quantities satisfy the relation $(\partial/\partial u)((\partial r/\partial u)/\xi) + \xi r_1 = 0$.

32. Professor Joseph Lipka: *Trajectory surfaces.*

In a space of any dimensionality V_n , any two directions through a point P determine a pencil of directions, and the curves through P in the pencil of directions which minimize the integral $\int \phi ds$, where ϕ is any point function and ds is the element of length, determine a surface or spread of two dimensions, called a trajectory surface (a generalization of a geodesic surface). In this paper the author discusses the Gaussian curvatures of these surfaces and their relations to the Gaussian curvatures of the corresponding geodesic surfaces. It is found, e.g., that a necessary and sufficient condition that the curvatures of a trajectory surface and its corresponding geodesic surface should be the same is that the determining pencil of directions contain the direction of the vector ϕ . Furthermore, we have here a generalization of Ricci's median curvature, principal directions, principal congruences, and principal invariants in any space V_n ; the theorems are analogous to those which Ricci has found in a discussion of geodesic surfaces.

33. Professor Joseph Lipka: *Geometric interpretation of the second differential parameter.*

Given any surface and any function ϕ of the coordinates on the surface, the invariant or second differential parameter of ϕ , usually designated by $\Delta_2\phi$, plays an important rôle in many problems in surface theory. The present paper contains a simple geometric interpretation of this parameter, which may be stated as follows: If any direction on the surface through a point P moves first by parallelism and then by conformal parallelism (whose characteristic function is ϕ) completely around an infinitesimal cycle drawn on the surface through P , the ratio of the angle formed by its final positions to the area of the cycle is the value at P of $\Delta_2\phi$.

34. Professor E. L. Dodd: *Formulas for the greatest and the least variate under general laws of frequency or error.*

The author classifies probability or relative frequency functions $\phi(x)$ as follows: For values of x sufficiently large, $\phi(x)$ is expressed as the product of an arbitrary $\psi(x)$, which satisfies certain inequalities, and (1) 0; (2) $x^{-1-\alpha}$; (3) g^{x^α} ; (4) $g^{(\log_e x)^\gamma}$; (5) g^{cx} ; (6) x^{-x} ; with $\alpha > 0$, $0 < g < 1$, $c > 1$, $\gamma > 1$; and then shows that for any positive η , there is for large enough n , a probability $> 1 - \eta$ that the greatest of n variates will be: (1) Constant, (2) $n^{1/\alpha}$; (3) $(-\log_g n)^{1/\alpha}$,

(4) c to the power $(-\log_g n)^{1/\gamma}$, (5) $\log_c (-\log_g n)$, (6) G in $G^G = n$, respectively; with a relative error small at pleasure for the values in (1), (3), (5), (6); and for $1/\alpha$ and $1/\gamma$ in (2) and (4). These results are applied to the seven Pearson types, the Bruns series (finite), the Jorgensen logarithmic function, the Poisson exponential (law of small numbers), the Charlier B -series (finite), and the Makeham life function. If G is determined from $\int_G^\infty \phi(t)dt = 1 - 2^{-1/n}$, it is, of course, equally probable that the greatest variate will or will not exceed G . This gives results for $ke^{-h^2x^2}$ in close agreement with those of Bortkiewicz.*

35. Professor T. R. Hollcroft: *Singularities that may be added to those of curves of given order.*

This paper will appear in an early issue of this BULLETIN.

36. Dr. S. D. Zeldin: *On the quadratic ternary partial differential equations admitting Lie groups of orders 4 and 5.*

The author determines the coefficients of the linear partial differential equation of the second order in three independent variables of the form $a_{11}z_{11} + a_{22}z_{22} + a_{33}z_{33} + 2a_{12}z_{12} + 2a_{13}z_{13} + 2a_{23}z_{23} + 2a_1z_1 + 2a_2z_2 + 2a_3z_3 + az = 0$ which is invariant under Lie groups of orders 4 and 5 of given structures.

37. Professor H. F. Blichfeldt: *On the approximate solution in integers of a set of m linear non-homogeneous equations in $n > m$ unknowns, and the final form of Kronecker's theorem.*

Kronecker's theorem asserts that if the coefficients of the equations in question satisfy certain "conditions of rationality," then a set of integers x_1, x_2, \dots, x_n exist which, when substituted for the unknowns, will satisfy the equations up to certain errors whose absolute values are all less than a previously assigned small positive quantity ϵ . In the present paper it is shown that if the errors are designated $\epsilon_1, \dots, \epsilon_m$, and if $R = \sqrt{x_1^2 + \dots + x_n^2}$, we may demand, in addition to $|\epsilon_i| < \epsilon$, also that $|\epsilon_1\epsilon_2 \dots \epsilon_m|R < f$, a certain number depending upon the coefficients of the given equations, but independent of ϵ and of x_1, \dots, x_n . When $m = 1$, f represents any preassigned positive number. It is furthermore shown that when $m > 1$, equations exist for which nothing may be demanded of the errors individually, beyond $|\epsilon_i| < \epsilon$; the case $m = 2, n > 3$ may furnish a possible exception.

* *Variationsbreite und mittlerer Fehler*, SITZUNGSBERICHTE DER BERLINER MATHEMATISCHE GESELLSCHAFT, vol. 21; October, 1921.

38. Professor Oswald Veblen and Mr. T. Y. Thomas
Geometries of paths admitting first integrals.

In this paper, the condition for the existence of homogeneous linear and quadratic integrals of paths in the geometry of paths introduced by Professors Eisenhart and Veblen (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), pp. 19–23) is reduced to the algebraic consistency of a set of equations, as is also the condition of existence of the general first integral subject to the particular restriction that its first covariant derivative vanish. Thus a necessary and sufficient condition for the geometry of paths to reduce to the Riemann geometry is given by the algebraic consistency of a set of equations. Other applications are also made. In addition the paper contains a discussion of normal coordinates for the geometry of paths from a different viewpoint from that adopted by Professor Veblen (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), pp. 192–197) and a treatment of the subject of covariant differentiation in the geometry of paths in which a whole group of tensors derivable from a given tensor is obtained. The first tensor of this group is the ordinary covariant derivative. The paper will appear in the TRANSACTIONS OF THIS SOCIETY.

39. Dr. T. H. Gronwall: *Isothermal surfaces with spherical lines of curvature in one system.*

By the use of pentaspherical coordinates, it is shown that all these surfaces are obtainable by inversion from the isothermal surfaces with plane lines of curvature in one system, which were investigated by Darboux (*Théorie des Surfaces*, vol. 4, chapter X). The present method determines all such surfaces; it yields Darboux's results in a simpler manner, and also a new class of surfaces of this kind with simple geometrical properties.

40. Dr. T. H. Gronwall: *Extension of Tchebychef's statistical theorem.*

In the first part of this paper, inequalities stronger than that of Tchebychef are found under the assumption that all moments of even order m_2, m_4, \dots, m_{2k} are known, k being any integer. No assumption on the probability function is made (except that it is never negative). In the second part, a similar investigation is carried out assuming the probability function monotone decreasing beyond the limit of skewness.

R. G. D. RICHARDSON,
Secretary.