BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

THE OCTOBER MEETING OF THE SOCIETY

The two hundred twenty-fourth regular meeting of the Society was held at Columbia University on Saturday, October 28, 1922, extending through the usual morning and afternoon sessions. The attendance included the following forty-eight members:

Alexander, Archibald, Barnum, Borden, Bowden, B. H. Camp, Cole, Cowley, Crum, L. D. Cummings, Douglas, Eisenhart, Fenn, Fields, Fine, Fite, Gill, Gronwall, Haskins, Hazlett, Hebbert, Hill, Hille, Joffe, Kasner, Lamond, Lamson, McDonnell, MacDuffee, MacNeish, Mathews, Mullins, Northcott, Pell, Pfeiffer, Reddick, R. G. D. Richardson, Ritt, Ruger, Seely, D. E. Smith, Sosnow, H. D. Thompson, Veblen, Weisner, H. S. White, Whited, Whittemore.

At the meeting of the Council, the following twenty-one persons were elected to membership in the Society:

Professor Margaret Buchanan, University of West Virginia;
Mr. Franz J. Feinler, Ladysmith, B. C.;
Mr. James Garfield Garrison, Polytechnic High School, San Francisco;
Mr. Charles Hopkins, University of Illinois;
Professor Daniel Hull, University of Notre Dame;
Mr. LeRoy Archibald MacColl, Western Electric Company;
Mr. Donald Hector MacPherson, Brown University;
Miss Frances Morrill Merriam, Wellesley College;
Mr. Edward Charles Molina, American Telephone and Telegraph Company;
Professor Paul Muehlman, Marquette University;
Miss Margaret Comstock Packer, Hood College;
Professor Llewellyn Rood Perkins, Middlebury College;
Mr. Emeterio Roa, University of Michigan;
Mr. William E. Roth, Phillips, Wis.;
Professor Hazel Edith Schoonmaker, Western College for Women;
Dr. Walter Andrew Shewhart, Western Electric Company;
Dr. Harry Melvin Shoemaker, North East High School, Philadelphia;
Professor John Theobald, Columbia College, Dubuque, Ia.;
Professor Clarence Eugene Van Horn, Judson College, Rangoon, Burma;
Mr. Wesley John Wagner, Purdue University;
Mr. Raymond Louis Wilder, University of Texas.

Twenty-two applications for membership in the Society were received.

A committee consisting of Mr. S. A. Joffe and Professor W. J. Berry was appointed to audit the accounts of the Treasurer for the current year, and to report to the Council at the Annual Meeting.

A list of nominations for officers and other members of the Council was presented by the Committee on Nominations, and was unanimously adopted by the Council At the head of this list was the nomination of Professor F. N. Cole for President of the Society. The Secretary reported that Professor Cole, while appreciating the honor done him, found himself unable to accept the nomination. The Council with regret accepted this decision, and adopted an alternative nomination presented by the Committee. The following resolution was adopted:

We, the Council of the American Mathematical Society, desire to place on record an expression of our profound regret that Professor Cole feels compelled to decline the nomination to the presidency of the Society. We believe that the members of the Society in general will share our disappointment that the opportunity is thus denied us to confer on Professor Cole the honor which would most suitably express our high esteem of him and of his signal services to the Society.

The Committee on the Cole Fund presented a report recommending that the fund be used to endow a prize to be called the Frank Nelson Cole Prize in Algebra. The recommendations, which are printed elsewhere in this BULLETIN, were accepted by the Council.

Professor J. K. Whittemore presided at the morning session of the Society, relieved in the afternoon by Professor C. N. Haskins. Titles and abstracts of the papers read at this meeting follow below. Mr. Linfield was introduced by Professor Birkhoff. The papers of Professor Jackson, Mr. Linfield, Mr. Rice, and Dr. Post, and Dr. Hille's third paper were read by title.

1. Professor Edward Kasner: *Parallels and geodesics in Weyl's affine geometry.*

The author first shows that the law of parallelism in a Weyl affine-connected manifold determines the geodesics (paths), but that the converse is not true. For the case $n = 4$, for example, 40 functions $\Gamma^{\alpha}_{\beta\gamma}$ (parallelism coefficients) determine the parallels, but 36 combinations (geodesic coefficients) determine the geodesics. Two sets of coefficients $\Gamma^{\alpha}_{\beta\gamma}$ have the same geodesics when they differ by $\frac{1}{2}(\delta_{\beta}{}^{\alpha}\, l_{\gamma} + \delta_{\gamma}{}^{\alpha}\, l_{\beta})$, where $\delta_{\beta}{}^{\alpha} = 1$ or 0 as $\alpha = \beta$ or $\alpha \neq \beta$, and l_1, l_2, l_3, l_4 are arbitrary functions. It is then shown if the geodesics in any Weyl manifold are represented, by means of an arbitrary point-to-point representation, on a euclidean space, they have a certain simple geometric property, called the *cubic property*, and that this is entirely characteristic. For the case of two dimensions the result is that the locus of the centers of curvature of the curves through a given point is a special cubic curve as stated in AMERICAN JOURNAL OF MATHEMATICS, vol. 28 (1906), pp. 207, 208. The extension to n dimensions now given is immediate, and, as remarked, is sufficient to characterize a Weyl family of geodesics or paths.

2. Professor Edward Kasner: *Einstein's equations of the second and third kinds.*

The three kinds of gravitational equations were introduced by Einstein in 1915, 1917, 1919 respectively. In the general form where the energy tensor $T_{\alpha\beta}$ appears in the right-hand member each kind is an actual extension of the previous kind. The present author gives a simple proof that if this tensor vanishes (space free from matter) the equations of second and third kinds are mathematically equivalent. An incidental result is that if in the cosmological equations $R_{\alpha\beta} - \lambda g_{\alpha\beta} = 0$ (sometimes called the De Sitter equations) we do not *assume* that λ is a constant, but merely assume that λ is a point-function, we can *prove* from the system itself that λ must be a constant. Thus we have a kind of analogue of Schur's famous theorem on spaces of constant Riemann curvature.

3. Professor Oswald Veblen: *Projective and affine geometry of paths.*

This paper contains a proof that any two affine geometries within the same projective geometry of paths are related by means of a vector and, conversely, that any affine geometry and a covariant vector determine another affine geometry. It will be published in the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES.

4. Dr. G. A. Pfeiffer: *Theorems on irreducible continua.*

The following theorems are proved in this paper: (I) If M is a bounded continuum which is irreducible between two points and C is a proper subcontinuum of M which contains one of these points, then $M - C$ is connected. (II) If M and N are bounded continua which are irreducible between the pairs of points a and b and b and c respectively, and if $M + N$ is irreducible between a and c, then any component of the product $M \cdot N$ which is not a point is either a continuum of condensation of $M + N$ or a non-decomposable continuum. (III) If M and N are bounded continua which are both irreducible between the same pair of points, then $M + N$ is a continuum which is not irreducible between any pair of points. (IV) If M is a continuum which is irreducible between the points a and b and C is a continuum of condensation of M which contains b, then M is irreducible between a and any point of C.

5. Dr. G. A. Pfeiffer: *On the mapping of dyadic sets.*

The writer shows that if A and B are two dyadic* sets and if A', a closed subset of A which is nowhere dense in A, and B', a closed subset of B which is nowhere dense in B, are homeomorphic, then there exists a (1–1) continuous correspondence between A and B which on A' is identical with any given (1–1) continuous correspondence between A' and B'. Some conditions on A' and B' other than mere homeomorphism between them is necessary for a theorem like the above.

6. Dr. Jesse Douglas: *On the analysis situs of the plane when the (directed) line is taken as element.*

The author has previously made use of a representation of the (directed) lines of the plane on a cylinder (this BULLETIN, vol. 28, p. 398). This correspondence being one–one and continuous, the analysis situs of the field of oriented lines of the plane is the same as that of the cylinder. There are two types of simple, continuous, closed curves in the plane, not equivalent under one–one continuous line transformation,

*See Hausdorff, *Grundzüge der Mengenlehre*, p. 322.

corresponding to curves surrounding or not surrounding the cylinder. The above is on the assumption of a metric plane. If the convention is made of a single line at infinity, the oriented lines of the plane have the analysis situs of the projective cone, the vertex corresponding to the line at infinity. If we postulate two lines at infinity, oriented clockwise and counter clockwise respectively, the analysis situs is that of the sphere. If ∞^1 lines are assumed at infinity, one parallel to each direction, the analysis situs is that of the torus.

7. Dr. Jesse Douglas: *Note on the integral of mean curvature over a surface.*

Let a polyhedron be inscribed in a portion Σ of a surface, and the number of its vertices increased indefinitely, while its faces grow smaller in such a way as to approach to tangent planes to the surface. Form the sum $\Sigma e_i A_i$, where e_i denotes any edge and A_i the infinitesimal dihedral angle of the faces meeting in that edge, A_i being counted as positive or negative according to a convention based on concavity and convexity considerations. The summation is to extend over all the edges of the polyhedron. Then the limit of the above sum is equal to

$$\int\int\left(\frac{1}{R_1}+\frac{1}{R_2}\right)d\sigma$$

taken over Σ. If the surface is minimal, this limit is equal to zero for every portion of the surface, and conversely. This suggests a possible approach to the solution of Plateau's problem, based on obtaining polyhedral approximations.

8. Professor Dunham Jackson: *Note on quartiles and allied measures.*

This paper appears in the present number of this BULLETIN.

9. Mr. B. Z. Linfield: *Particle geometry.*

After setting up a set of postulates for the general n-dimensional particle varieties, the author considers primarily the dense and normal varieties, U_n and S_n respectively. For them he proves a number of separation theorems which finally lead to the definition of the connectivity of an S_n. He concludes with a number of theorems concerning the S_2 of lowest connectivity, pointing out that the postulates for an S_2 of lowest connectivity form a complete set of axioms for the four-color problem and that thus the theorems concerning the S_2 of this connectivity are immediately transferable to theorems in the four-color problem.

10. Mr. B. Z. Linfield: *On certain polar curves with applications to the location of the zeros of the pth derivative of a rational function.*

The process of polarization is here used on rational fractional functions of three homogeneous variables. After some of the properties of these curves are established the results are applied to the generalized Van den Berg curves,

$$\partial^r(ux_i + vy_i + w)^{\mu_i}/\partial w^r = 0.$$

A further study yields results about the zeros of the pth derivative of a rational function from which the root-polygon theorem follows as a particular case.

11. Mr. L. H. Rice: *On the expression of the sum of any two determinants as a determinant of more dimensions.*

In this paper it is shown that the sum of any two determinants A and B of the same number of dimensions, the same signancy, and the same order, can be expressed as a determinant C of the next higher class (number of dimensions). The sum is so formed that the elements of C are of a very simple character, being in fact identical with individual elements of A and B. It appears that upon changing the sign of every element of a certain sublayer of C, the signancy of C may be varied. This paper appears in the JOURNAL OF MATHEMATICS AND PHYSICS OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY, vol. 1, No. 3.

12. Dr. Einar Hille: *A Pythagorean functional equation.*

The author proves that the general solution of the functional equation

$$|f(z)|^2 = |f(x)|^2 + |f(iy)|^2; \qquad z = x + iy,$$

is given by $c \sin az$ where c is an arbitrary constant and a is constant, either real or purely imaginary. The special case $c = 1/a$, $a \to 0$ yields a particular solution, namely z itself. The proof consists in using the fact that $\log |f(x + iy)|$ is an harmonic function the Laplacian of which is zero. The resulting differential equations are integrable by quadratures. The method is easily extended to similar functional equations.

13. Dr. Einar Hille: *A class of functional equations.* Preliminary communication.

The problem here considered is to determine the analytic solutions of the functional equation

$$w = R(u, v); \quad w = |f(x+iy)|; \quad u = |f(x)|; \quad v = |f(iy)|,$$

where x and y are real variables and R is a rational function of u and v, satisfying certain conditions of symmetry and reality. Observing that the Laplacian of log R must vanish identically, we obtain a differential equation of the form $G_1u'' + G_2(u')^2 + G_3v'' + G_4(v')^2 = 0$ where G stands for a rational function of u and v with constant coefficients. Keeping y (or x) constant, u (or v) can be determined by three quadratures. If this process gives an analytic solution, $f(z)$, of the functional equation that is *single-valued* throughout the plane, then $f(z)$ is an *elliptic function* with one real and one pure imaginary period, or else a degenerate case of such a function, or, finally, a function of the form exp. $(az^2 + bz)$. The classification of the many-valued solutions is not yet completed. The method is easily extended to the case when R is an algebraic function of u and v.

14. Dr. Einar Hille: *Oscillation theorems in the complex domain.*

This paper is devoted to the study of the distribution in the complex plane of the zeros of solutions of a linear homogeneous differential equation of the second order, the coefficients of which are analytic functions of a complex variable. An integral equality, corresponding to Green's formula in the real theory, is deduced, by means of which it is possible to determine regions in the complex plane in which a given solution is different from zero as well as its first derivative. Various types of such regions are considered. The paper also contains a study of the asymptotic distribution of the zeros of a solution in the neighborhood of an irregular singular point of rather general type. The paper will appear in the TRANSACTIONS OF THIS SOCIETY.

15. Professor W. L. Crum: *Note on the internal evidence of the reliability of a test.*

The paper seeks to discover the theoretical limitations on the use of the common method of computing a reliability coefficient by comparison of the results for the odd- and evennumbered questions of the test. It is shown that this method is strictly applicable only to tests which are known to have very special degrees of uniformity, and that it is likely to lead to altogether inaccurate conclusions as to reliability for the irregular tests which we commonly give.

16. Professor W. L. Crum: *The use of the median in determining indices of seasonal variation.*

The paper suggests that the distribution of the monthly variables, in the usual historical series, about their respective averages has such a form that the arithmetic mean is a less reliable measure of the average tendency than is the median. A study is made of a particular historical economic series, known to possess marked and fairly regular seasonal variation, with a view to verifying the hypothesis. The conclusion appears unmistakable that the median should be used.

17. Professor R. M. Mathews: *A general construction for circular cubics.*

Let Q and R be two points on a variable circle which is cut again in S and T by two arbitrary fixed lines s and t through Q and R, respectively. The variable line ST cuts an arbitrary fixed line g in U. The variable line PU, drawn from a fixed point P, cuts the circle in A and B. The locus of these two points for the pencil of circles through Q and R is a circular cubic curve. Conversely, every circular cubic may be constructed in this manner.

18. Professor R. M. Mathews: *A theorem on conics, with applications.*

Three conics of the coaxial set $\{K\}$ on two points R and S cut an arbitrary fixed conic C in three sets of four points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$, $(i = 1, 2, 3, 4)$, respectively; the conic of the set through $A_iB_iC_i$ cuts C in a fourth point D_i. Then the set of four points $\{D_i\}$ lies on a conic through R and S.

When R and S are the circular points at infinity the set $\{K\}$ consists of the circles of the plane; when the figure now is inverted we obtain theorems about groups of concyclic points on circular cubics and on bicircular quartics.

19. Mr. Louis Weisner: *A property of the characteristic elements of a group.*

If a group G can be generated by an invariant element S

and an invariant subgroup H, there exists an automorphism of G in which each element of H corresponds to itself and S corresponds to S^i, where i is a certain integer different from unity and prime to the order of S. The only exception to this statement is the case where the order of S is twice an odd number and H is of index 2 in G. It follows that a characteristic element of G whose order is not twice an odd number cannot appear in any possible set of independent generators of G, and is therefore contained in the ϕ-subgroup of G.

20. Dr. E. L. Post: *Visual intuition in Lobachevsky space.*

According to Klein, the development of non-euclidean geometry has proceeded through three stages, the synthetic stage centering around Lobachevsky, the differential geometry stage initiated by Riemann, and the projective measurement stage developed by Cayley and Klein. The present paper may be said to belong to a fourth stage, already found in the work of Poincaré. It takes an observer brought up in euclidean space, immerses him in Lobachevsky space, and relates what he sees there. In particular, the observer views the Lobachevsky straight line from all positions, and notes that in general it has the appearance of one branch of a hyperbola, always spreading away from him, and varying in size and shape with his distance from the line. He then observes objects at various distances from him, and notes that for a given distance an object appears much smaller in Lobachevsky than in euclidean space. Another interpretation is that Lobachevsky space is much roomier than euclidean space as one goes out from a given fixed center. These two developments of the Lobachevsky intuition are then related by showing that the shortest line between two points as determined by the metric presents the appearances described above.

21. Professor Elizabeth B. Cowley: *Note on a generalization of the old puzzle of* 8, 5, *and* 3 *pint vessels.*

Bachet (*Problèmes Plaisants & Delèctables,* 5th edition, Paris, 1884) has a discussion of a generalization of the old puzzle to obtain 4 pints of liquid from an 8 pint vessel full of liquid by the use of 2 empty vessels with capacities of 5 and 3 pints. Attention is called to the case in which $A < (B + C)$ (where A, B, C represent the capacities of the largest, middle, and smallest vessels respectively). Two examples are given: 20, 13, 9 and 16, 12, 7, and the statement is made that the first is possible and the second impossible but that these facts are not known *a priori*. In this paper, criteria are worked out for ascertaining *a priori* whether a solution is impossible when $A < (B + C)$. Incidentally, some properties of cases in which $A = B + C$ are noted.

R. G. D. RICHARDSON,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION

The fortieth regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of California, October 21, 1922. Professor Allardice presided during the early part of the meeting and Professor Cajori during the latter part. The total attendance was thirty-five, including the following twenty-four members of the Society:

Alderton, Allardice, Andrews, Barter, Bernstein, Blichfeldt, Buck, Cajori, Edwards, Growe, Haskell, Hoskins, Irwin, Lehmer, Levy, Libby, Moreno, F. R. Morris, T. M. Putnam, Shane, P. Sperry, Stromquist, A. R. Williams, Wong.

The following officers were chosen for the year: Chairman, Professor Florian Cajori; Secretary, Professor B. A. Bernstein; Programme Committee, Professors H. F. Blichfeldt, D. N. Lehmer, B. A. Bernstein.

It was decided to hold the next Fall meeting of the Section on Saturday, October 20, 1923, at the University of California.

Titles and abstracts of papers read at this meeting follow. The papers of Professors Bell and Smail were read by title.

1. Dr. J. D. Barter: *Spiral functions.* Preliminary report.

If α is a root of the equation $a_0 + a_1x + a_2x^2 + \cdots + \lambda + a_nx^n = 0$, termed the "fundamental equation," then exp. $(\alpha t) = f_0(t) + \alpha f_1(t) + \alpha^2 f_2(t) + \cdots + \lambda + \alpha^{n-1}f_{n-1}(t)$. The functions f are termed spiral functions. Their general properties and applications are summarized. Methods for calculating the values of these functions in special cases are outlined.