NOTE ON STEADY FLUID MOTION

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In a previous paper* I have shown how to find special invariant configurations of the projective and linearoid groups investigated by Wilczynski† in connection with steady fluid motion. It is the purpose of this note to show how the group whose general infinitesimal transformation is

$$Kf = u(x)\frac{\partial f}{\partial x} + v(x, y)\frac{\partial f}{\partial y} + w(x, y, z)\frac{\partial f}{\partial z}\,‡$$

should be simplified in order to represent the steady motion of a fluid under the influence of forces possessing a potential.

If the external forces have a potential, then, as is known, the functions u, v, w must be such that the expression

$$Kudx + Kvdy + Kwdz$$

is a complete differential, or, what amounts to the same thing,

$$(1) \qquad \begin{aligned} \frac{\partial Ku}{\partial z} - \frac{\partial Kw}{\partial y} &= 0, \\ \frac{\partial Kw}{\partial x} - \frac{\partial Ku}{\partial z} &= 0, \\ \frac{\partial Ku}{\partial y} - \frac{\partial Kv}{\partial x} &= 0. \end{aligned}$$

Performing the operations indicated by equations (1) we get:

$$(a) \qquad \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} = 0,$$

$$(b) \qquad \frac{\partial u}{\partial x}\cdot\frac{\partial w}{\partial x} + u\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial x\partial y} + w\frac{\partial^2 w}{\partial x\partial z} + \frac{\partial v}{\partial x}\cdot\frac{\partial w}{\partial y} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial z} = 0,$$

$$(c) \qquad u\frac{\partial^2 w}{\partial x\partial y} + v\frac{\partial^2 w}{\partial y^2} + w\frac{\partial^2 w}{\partial y\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z} = 0,$$

* JOURNAL OF MATHEMATICS AND PHYSICS (Mass. Inst. of Tech.), vol. 1 (1921), p. 54.

† TRANSACTIONS OF THIS SOCIETY, vol. 1 (1900), pp. 339–352.

‡ This class of groups has been investigated by Sophus Lie in connection with two-point invariants. See Lie-Engel, *Theorie der Transformationsgruppen*, vol. 3, Abtheilung 5.

which can be written in the form

$$(a)\qquad \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right)=0,$$

$$(b)\qquad \frac{\partial}{\partial x}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=0,$$

$$(c)\qquad \frac{\partial}{\partial y}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=0.$$

From (a) follows at once that Kv *can be a function of* y *only.* From (b) we see that Kw must be a function of y and z alone, and, since by virtue of (c) Kw must be a function of x and z alone, it follows that Kw *can be a function of* z *only.*

If the fluid is incompressible, we have the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

and therefore w is a *linear function* of z.

In the case of irrotational motion, since

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0,\qquad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0,\qquad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0,$$

we must have, in the above infinitesimal transformation Kf, u *a function of* x *alone,* v *a function of* y *alone,* and w *a function of* z *alone.* There exists then a velocity potential, say F, where

$$u=\frac{\partial F}{\partial x},\qquad v=\frac{\partial F}{\partial y},\qquad w=\frac{\partial F}{\partial z},$$

and the orthogonal trajectories of the family of surfaces

$$F=\int u(x)dx+\int v(y)dy+\int w(z)dz=\text{constant}$$

represent the stream lines. These stream lines are the intersections of the two families of cylinders obtained by solving the equations

$$\frac{dx}{u(x)}=\frac{dy}{v(y)}=\frac{dz}{w(z)}\,.$$

The separately invariant points will be found by solving the equations

$$u(x)=0,\qquad v(y)=0,\qquad w(z)=0.$$

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