BOOKS ON RELATIVITY

Das Relativitätsprinzip. Lorentz. Einstein. Minkowski. Fortschritte der mathematischen Wissenschaften in Monographien. Herausgegeben von O. Blumenthal, No. 2. Leipzig und Berlin, B. G. Teubner, dritte Auflage, 1920. i + 146 pp.

Raum. Zeit. Materie. Von Hermann Weyl. Berlin, Julius Springer, vierte Auflage, 1921. Mit 15 Textfiguren. ix + 300 pp.

Relativity. The special and the general Theory. By Albert Einstein. Translated by Robert W. Lawson. New York, Henry Holt and Co., 1921. Frontispiece. xiii + 168 pp.

The Theory of Relativity. By Robert D. Carmichael. Mathematical Monographs, Edited by Mansfield Merriman and Robert S. Woodward, No. 12. New York, John Wiley and Sons, 2nd edition, 1920. 112 pp.

Das Relativitätsprinzip. Leichtfasslich entwickelt von Adam Angerbach. Leipzig und Berlin, B. G. Teubner, 1920. Mit 9 Figuren im Text. 57 pp.

The Concept of Nature. Tarner Lectures delivered in Trinity College, November, 1919. By A. N. Whitehead. Cambridge, The University Press, 1920. viii + 202 pp.

Wiskunde, Waarheid, Werkelijkheid. Door L. E. J. Brouwer. Groningen, P. Noordhoff, 1919. 12 pp. + 23 pp. + 29 pp.

For scientists generally, and especially for mathematicians and physicists, who understand best many of the questions involved, the theory of relativity has fundamental interest. In the following pages our purpose is to pass in review the above recent books dealing with the theory and at the same time to indicate its present state and some unsolved problems.

The collection of monographs gathered by Blumenthal begins with two papers by the Dutch physicist, Lorentz, the second and more important one of which appeared in 1904. By endeavoring to unite the classical Newtonian mechanics and the electromagnetic theory of Faraday and Maxwell into a single consistent theory, one is necessarily led to absolute space (the ether) and absolute time. In fact, physics has stood committed to absolute time since the acceptance of Newton's law of gravitation. But the experiments of Michelson in 1881 yielded an opposing result. Lorentz, in common with other physicists, had the conviction that the universe was electromagnetic in character, and he turned to the electromagnetic equations for an explanation of the difficulty. His answer to the apparent contradiction of theory and experiment was based upon the fact that the equations admitted of a transformation in which space and time were intermingled. On this basis, without giving up the concepts of absolute space and time, he was able to explain the paradox by assuming that bodies undergo a slight contraction in the direction of their motion, which for the earth is not more than a few inches. To an observer moving at uniform velocity, the same electromagnetic equations appear to hold because such an observer uses "local time." Lorentz's explanation violated a fundamental principle, namely the pragmatic principle that no physical entity exists if its presence can never be determined by any conceivable experiment. Absolute space and time are this type of entity in his theory.

A year later in 1905, but independently, Einstein wrote his paper *Zur Elektrodynamik bewegter Körper*, which is the third paper of the collection. In this he lays the foundation of the so-called special theory of relativity. Einstein starts with a peculiarly simple type of physical universe, perhaps the simplest in harmony with all known physical laws. This is the universe of empty isotropic space in which there are infinitesimal inertia particles. The particles *appear* from such a particle to move with uniform velocity in a straight line, if observations of light signals are made with the aid of a clock. Thus the fundamental measuring instrument is the clock. It is further assumed that a light pulse appears to advance at a constant velocity from any such particle (the Michelson experiment). On the basis of these postulates the transformation equations between the coordinates set up from reference particles are deduced, and it is shown that the Maxwell electromagnetic equations are unaltered under precisely this group (the Lorentz group) of transformations. The behavior of the electron as experimentally determined is in conformity with this theory. In the short paper that follows Einstein notes that the same discussion indicates that the apparent mass of a system will depend upon its energy.

The fifth article of the collection is the mathematician Minkowski's remarkable *Raum und Zeit* of 1908. In this article, which threw a flood of light upon the work of Einstein and Lorentz, the geometry of four dimensions furnished the principal weapon. If Minkowski had lived, doubtless other equally important contributions to the theory of relativity would have come from his pen, and in any case it is clear that his influence upon Einstein can scarcely be overestimated.

The gist of Minkowski's paper is as follows: In the four-dimensional relativistic manifold of space and time, a pair of world-points or events are associated with a unique number, namely, if a particle move from the earlier world-point to the later world-point, the interval of local time elapsed will give this number. The mathematician will realize at once that we have here the elements of a non-euclidean geometry of four dimensions of simple type. The Lorentz transformations are merely the transformations of the geometry which leave this interval between world-points unaltered, and the whole theory may be subsumed in the single equation

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2,$$

where ds is the local time element, c is the velocity of light, and dx, dy, dz, dt have their customary meanings. In the same article it is pointed out how the laws of motion of the electron may be interpreted in this space; and a suitable modification of the Newtonian law of attraction is made, such as Poincaré had given earlier. There follow some instructive notes by the mathematical physicist Sommerfeld.

The remaining articles of the collection are reprints of more recent articles by Einstein, and four of them do not appear in the earlier editions. It is *Die Grundlagen der allgemeinen Relativitätstheorie* of 1916 which has aroused such widespread attention. Concerning it, Sommerfeld says in the notes just mentioned: "This general relativity theory is logically so unified and satisfactory that it has found unconditional acceptance, especially in mathematical quarters." In a paragraph added after learning of the verification of Einstein's quantitative prediction of the deviation of light by the sun, Sommerfeld says further, "The general relativity theory can therefore be regarded as an *established proposition.*" If this is the truth, physical science is entering upon an era in which the new view will differ radically from the classical one.

To the mathematician, Einstein's generalized theory is of interest in several respects. In the first place it illustrates afresh the importance of taking the simplest possible case as an abstract basis of departure. Secondly, Einstein uses mathematical analogy in passing, step by step, from the simple universe of the special theory to the most general universe, and at each step the mere sense of mathematical form is sufficient to point toward a natural generalization. The mathematician may feel satisfied that the formal analogies supplied by classical dynamics and four-dimensional geometry furnish the very basis by which Einstein's generalization proceeds. And, thirdly, the technical tool which made elaboration and verification of the theory possible is the invention of the mathematicians Riemann, Christoffel, and more especially of Ricci and Levi-Civita—namely the absolute differential calculus.

What then are these successive steps of Einstein? To the writer they appear as follows:

(1) In the special theory of relativity, the universe consists of an empty isotropic space with infinitesimal inertia particles, and the central formula is that for ds^2 given above.

(2) A somewhat more general type of universe is that of a non-isotropic empty space formed by a gravitational field. It is natural, by analogy with the Riemann geometry, to assume that local time is given by a quadratic differential form ds^2 in the space and time variables, and that the particular coordinates chosen are irrelevant (general theory of relativity). But in such case only six of the ten coefficients in ds^2 must be regarded as arbitrary. Therefore there are required six equations to fix these coefficients and these conditions must be independent of the coordinate system. This leads to Einstein's conclusion that the contracted Riemann tensor vanishes. By analogy with the special theory of relativity, the paths of the particles appear as the geodesics and the paths of the light pulse satisfy the equation $ds = 0$.

(3) Still more generality is obtained if matter and energy are present. For case (1), this leads to the vanishing of the divergence of an "energy tensor." In the equations obtained in case (2) the left-hand members are tensors while the right-hand members vanish. It is natural then to assume by analogy with classical dynamics that, in case the space contains matter and energy, the right-hand member becomes the energy tensor. If we

assume this to be the case, the complete equations, as general in their scope as those of the classical theory, are obtained.

The causal principle is effective, but in an obscure form, as follows: At any "instant" for the coordinate system under consideration the time rate of change of the derivatives of the gravitational tensor formed by the system of coefficients in ds^2 and of the energy tensor are known. Also the apparent accelerations of the particles are thereby determined. Consequently it is possible to obtain the new value of the tensors and their derivatives, and the new positions and velocities of the particles, an instant later, and so to proceed indefinitely.

According to Einstein the acceleration of a particle in empty space is due to gravitational forces, and depends only on ds^2 and the coordinates chosen. The law of motion at low velocities is found to be the same as that of Newton except for very small modifications. To make a specific application, Einstein determines the necessary form of ds^2 for a single central body such as the sun, also on a postulational basis, and arrives at his brilliant predictions.

In the final paper of the collection, written in 1919, Einstein shows how the idea of a spherical space may be conveniently introduced to eliminate the difficulties due to boundary conditions.

The casual reader of the theory of relativity will feel a certain lack of concreteness. The classical physical theories seemed to touch reality in at least three ways, namely in the independent concepts of space, of time, of force. I take it to be self-evident that any genuine physical theory must touch reality somewhere. So far as I can see, the Einstein theory does this at one and at only one place, namely in its concept of local time which can be measured by means of the natural clock, the atom. In an article appearing in January, 1921, in the PROCEEDINGS OF THE BERLIN ACADEMY, Einstein lays emphasis upon this notion of the natural clock as the fundamental element, but one could wish that this had been done more definitely in his earlier articles.

From the scientific point of view the most important of the other books which we desire to review is Weyl's *Raum. Zeit. Materie.* The Einstein theory has a certain pliability in the presence of an energy tensor, which may be modified to suit the exigencies of the physical situation under discussion. On the other hand, this pliability will appear as a defect to some minds since it provides physical science with a blank form rather than with a definitive theory. It may naturally be expected that theories will be forthcoming which attempt to explain non-gravitational phenomena also on a similar quasi-geometrical basis. One recalls here the vortex theory of the atom as an analogous attempt in classical physics.

The original part of Weyl's valuable and complete treatise consists in an attempt to deduce the electromagnetic equations in such a manner. For this purpose he invents a generalization of the Riemann geometry. In the Riemann geometry, the elements ds^2 can be compared at various parts of the manifold. Weyl notes that this is a species of action at a distance and proposes to compare the elements ds^2 only for various directions at a world-point. In other words his quadratic form ds^2 is one in which merely the ratios of the coefficients are important and the coefficients appear as undetermined up to an arbitrary multiplicative scale-factor.*

Thus in the normalization of his quadratic form by change of variables, he has five arbitrary functions (the four arbitrary coordinate functions and his scale-function) instead of the four functions available in the Einstein theory. Weyl is able to use the notion of parallel displacement due to Levi-Civita; namely, the small vector can be displaced so as to maintain size and direction in a specially chosen *geodesic* coordinate system. As this vector varies in position and returns to its starting point it will not have the same length except in the case of the Riemann geometry. The logarithmic derivative of the scale-function is a differential, whose four components behave as the components of an electromagnetic potential.

An obvious objection to Weyl's extension is that he loses contact with the real, for ds can no longer stand for the element of local time; otherwise we should expect atoms of the same element with different past histories to have different rates of vibration, and such has never been observed to be the case. In the second place, a modified ds^2 for the same manifold can be obtained which is invariant as in the Riemann geometry, and thus we are led back to the Einstein theory together with a single independent equation of the type coming under Einstein's theory. These criticisms of Weyl's work are given in the March, 1921, number of the PROCEEDINGS OF THE BERLIN ACADEMY by Einstein. Eddington has proposed a further modification of Weyl's theory in the April number of the PROCEEDINGS OF THE ROYAL SOCIETY. Eisenhart and Veblen have gone much further in an important paper in the PROCEEDINGS OF THE NATIONAL ACADEMY, February, 1922.

We pass now to the more popular treatments given in the next three books. The first of these is by Einstein himself, and affords an interesting and skillful approach to the fundamentals of the theory.

The second edition of Carmichael's book contains his earlier treatment of the special theory based upon a set of physical postulates. The new chapters are a direct summary of the results of the general theory, as presented by Einstein and Eddington. This summary is too abbreviated to be followed with much profit by the reader who has not delved elsewhere into the theory.

The tiny pamphlet by Angersbach presents a brief historical development of the notions underlying the special relativity together with an elementary presentation. It is readable.

It is obvious that the relativity theory has decided significance for philosophical thought, as indeed every new physical theory must have. In his interesting book, *The Concept of Nature*, the English philosopher-mathematician Whitehead expounds his views of the physical universe in the light of the theory of relativity. The book has obvious relations with an earlier book. † The main idea of Whitehead is that the underlying

* See his recent papers in the MATHEMATISCHE ZEITSCHRIFT, vol. 12, Nos. 1, 2 (1922).

† *An Inquiry Concerning the Principles of Natural Knowledge*, A. N. Whitehead. Cambridge, University Press, 1919. xii + 200 pp.

realities are attained by a method of extensive abstraction, a spatial point for instance being generated by all the objects of a certain category (those which include it spatially). His analysis of experience is very interesting. The mathematician will regret frequently redundancy and vagueness in philosophical treatises. Of this there is little in Whitehead's book. The importance and the exactitude of many of his analyses must be admitted. Maxime Bôcher once said to the writer, "What man would be a philosopher who might be a mathematician!" One feels that Mr. Whitehead deserves both titles.

The contrasting account given in this book between the old theories and the new theory of relativity is interesting. Characterizing the old theories, Whitehead says: "For example, colour is the result of a transmission from the material object to the perceiver's eye; and what is thus transmitted is not colour. Thus colour is not part of the reality of the material object. Similarly for the same reason sounds evaporate from nature. Also warmth is due to the transfer of something which is not temperature. Thus we are left with spatio-temporal positions, and what I may term the 'pushiness' of the body. This leads us to eighteenth and nineteenth century materialism, namely, the belief that what is real in nature is matter, in time and in space and with inertia."

The new relativity theory he expounds as follows: "Let us make therefore the general statement that four measurements, respectively of independent types (such as measurements of lengths in three directions and a time) can be found such that a definite event-particle is determined by them in its relations to other parts of the manifold. . . . If (p_1, p_2, p_3, p_4) be a set of measurements of this system, then the event-particle which is thus determined will be said to have p_1, p_2, p_3, p_4 as its coordinates in this system of measurement." . . . "Then we should naturally say that (p_1, p_2, p_3) determined a point in space and that the event particle happened at that point at the time p_4. . . . Furthermore the inhabitant of Mars determines event-particles by another system of measurements. Call his system the q-system. According to him, (q_1, q_2, q_3, q_4) determines an event-particle, and (q_1, q_2, q_3) determines a point and q_4 a time. But the collection of event-particles which he thinks of as a point is entirely different from any such collection which the man on earth thinks of as a point. Thus the q-space for the man on Mars is quite different from the p-space for the land-surveyor on earth. . . ."

". . . We have got to find the way of expressing the field of activity of events in the neighborhood of some definite event-particle E of the four-dimensional manifold. I bring in a fundamental physical idea which I call the 'impetus' to express this physical field. The event-particle E is related to any other neighboring event-particle P, by an element of impetus." ··· "Einstein showed how to express the characters of the assemblage of elements of impetus of the field surrounding an event-particle E in terms of ten quantities which I will call J_{11}, J_{12}, . . . The numerical values of the J's will depend on the system of measurement adopted, but are so adjusted to each particular system that the same value is obtained for the element of impetus between E and P, whatever be the system of measurement adopted. This fact is expressed by saying that the ten J's form a 'tensor.' " . . . "We now return to the path of the attracted particle. We add up all the elements of impetus in the whole path, and obtain thereby what I call the 'integral impetus.' The characteristic of the actual path as compared with neighbouring alternative paths is that in the actual paths the integral impetus would neither gain nor lose, if the particle wobbled out of it into a small extremely near alternative path."

Evidently Whitehead is expressing the relativistic theory of the path of a particle in a gravitational field. The indefinite general terms used stand in unfavorable contrast with those which can be used in the exposition of the classical theory.

Finally we turn to the little pamphlet by Brouwer. Many American mathematicians have read in this BULLETIN (November, 1913) a translation of Brouwer's paper on *Intuitionism and Formalism*, which is the final essay of the pamphlet. Those who know the mathematical work of Brouwer will be interested in these essays, in the second of which he touches upon the special theory of relativity with emphasis upon the notion of *group*. Attention should also be directed to his noteworthy analysis of the logical principle of the *excluded middle*, to which reference is made in the first essay.

In conclusion, one or two remarks of general character suggest themselves.

The theory of relativity in its general form or in its special form involves a definite group. To the mathematician at least it would be of considerable interest to see physical theories developed for other special groups. In particular the most general group of all, that of *analysis situs*, suggests itself, for this alone appears strictly proper in the general theory of relativity, where any transformation of coordinates whatsoever ought to be admitted. If such a theory can be constructed, the interrelation of continuous manifolds of arbitrary form will form the essential element. A theory of this kind would seem to be consonant with quantum theory.

Also, with others, we may call attention to the fact that no theory of relativity so far explains the difference between positive and negative electricity, or throws any light upon the constitution of matter. Furthermore no real reason appears why the velocities of the stars relative to one another are so small in comparison with the velocity of light.

While awaiting further developments, let us at least say with Whitehead of Einstein's investigations: "They have made us think." Some may agree with the final words of Weyl: "A few chords of that harmony of the spheres of which Pythagoras and Kepler dreamed have fallen upon our ears."

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