ON KAKEYA'S MINIMUM AREA PROBLEM*

BY W. B. FORD

1. *Introduction.* During recent years the Japanese school of mathematicians, notably Professors Hayashi, Kakeya and Fujiwara, have proposed and investigated to some extent a unique and apparently new class of maxima-minima problems of which the one considered in this paper may be regarded as the simplest type. In general, such problems concern the determination of the closed curve of least area within which a given configuration may be completely rotated. The special problem in which we shall be interested appears to have been first stated by Kakeya and is as follows:†

A line-segment AB lying in the plane MN is to be moved so that it shall return to its original position but with its ends reversed (as in the rotation of a segment about its middle point through a semicircumference). How should this be done in order that the area generated during the motion may be a minimum?

2. *Interpretations of the Problem.* As thus stated, we note first that the problem admits of the following two interpretations: In computing area generated during any portion of the motion, the area S bounded by any given enclosure in the plane MN is to be counted (*a*) as many times as it is passed over by AB; (*b*) never more than once.

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† The existing literature upon this and the more general problems above referred to appears to be chiefly confined to the following three papers: *On the curves of constant breadth, and the convex closed curves inscribable and revolvable in a regular polygon*, by Tsuruichi Hayashi, Tôhoku Science Reports, vol. 5, pp. 303–312 (Dec. 1916); *On some problems of maxima and minima for the curve of constant breadth and the in-revolvable curve of the equilateral triangle*, by M. Fujiwara and S. Kakeya, Tôhoku Journal, vol. 11, pp. 92–110 (Feb. 1917); *Some problems on maxima and minima regarding ovals*, by Soichi Kakeya, Tôhoku Science Reports, vol. 6, pp. 71–88 (July, 1917). Recently Pál (Mathematische Annalen, vol. 83 (1921), pp. 311–319) has given a complete solution, but under greater restrictions than those of this paper.

These two interpretations are illustrated in Fig. 1 wherein AB has been given a simple rotation of angle θ about a fixed point O lying in the perpendicular bisector CD of AB. Here A and B describe arcs of one and the same circle, thus taking the final positions A', B', while the segment AB passes over the singly shaded areas R, T once each, but passes over the doubly shaded area S twice, first by the portion CB and later by CA. Hence, for such a motion the area generated according to interpretation (a) is $R + 2S + T$, while in interpretation (b) it is $R + S + T$.

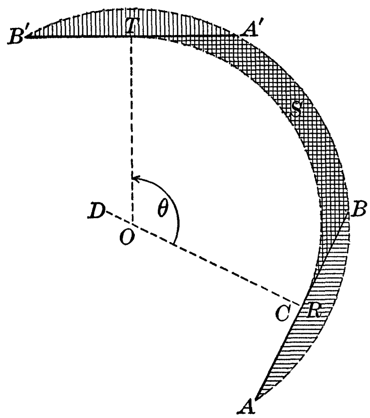

FIG. 1.

In order to make the necessary distinction thus arising, we shall hereafter refer to area generated in the sense (a) as area swept *over*, and to that generated in the sense (b) as area swept *out*.* We proceed, therefore, to consider the problem under interpretation (a) and it is believed that the method followed leads to a complete solution in this case.

3. *Infinitesimal Rotation.* Instead of undertaking directly the problem of § 1 wherein AB is to be turned completely end for end and finally brought back upon itself, we shall find it desirable to begin by considering the following more general yet in some respects more simple question.

How should a line-segment of length $2l$ be moved in such a way that the angle between its initial and final directions shall be a given amount, θ, while the area swept *over* (§ 2) shall be a minimum? Thus, suppose that AB is the initial position, its direction being regarded as from A to B, and let

* In the studies of Kakeya and others already referred to, the term "area generated" is taken in the sense (b) only, this being the case of greatest complexity and interest, but inasmuch as our method for the study of (b) depends essentially upon that for the more simple case (a), we shall find it desirable to develop the latter first.

CD be a line whose direction (from C to D) makes the given angle θ with AB. The question then is, how should AB be given the same direction as CD in such a way as to sweep over a minimum area?* This question may be answered directly by use of the following simple kinematical principle which, for brevity, we shall assume without proof.

If a line-segment AB of length $2l$ lying in a plane MN is given an infinitesimal rotation of angle $d\theta$ about a fixed point O in MN, the area swept over will be less when O lies upon the perpendicular bisector of AB than when it lies at any point elsewhere.†

It thus appears, as regards the question proposed above, that for the desired minimum it is necessary and sufficient that during the motion each infinitesimal rotation shall be about some point in the perpendicular bisector of the segment. In fact, the infinitesimal area then swept over by an increment $d\theta$ in direction will be less than that obtained upon any other plan yielding the same change in direction, hence the same will be true of the sum of such infinitesimal areas and likewise of the limit of this sum, which limit is the area in question. In the customary language of kinematics, this means that the instantaneous center of motion should lie at all times upon the perpendicular bisector.

Moreover, since each infinitesimal area corresponding to a change in direction thus comes to differ from the value $l^2 d\theta$ by an infinitesimal of higher order than the first as compared to $d\theta$ as readily appears, it follows by Duhamel's theorem that the minimum area itself will have the value

$$l^2 \int_0^\theta d\theta = l^2\theta.$$

* We assume throughout that during the motion the angle which the segment makes with its initial direction increases only monotonically, as otherwise negative areas would be generated. However, θ is not restricted to the range $0 < \theta \leqq 2\pi$, but may be assigned any positive value whatever.

† The proof is readily carried out. In case the rotation takes place about a point in the perpendicular bisector the area swept over differs by an infinitesimal of higher order than $d\theta$ from $l^2 d\theta$, while if the rotation takes place about a point whose distance is h $(h > 0)$ from the perpendicular bisector, the area swept over differs by an infinitesimal of higher order than $d\theta$ from $(l^2 + h^2)d\theta$.

Evidently there are an infinite number of ways of the type just described for moving AB into parallelism with CD. Of these the simplest is that in which AB is given a pure rotation of angle θ about its middle point regarded as fixed. The next simplest case is that in which AB is given a pure rotation of angle θ about some point in its perpendicular bisector other than its own middle point, thus sweeping over a circular strip such as shown in Fig. 1. However, it is to be observed that in general a movement such as we are considering will consist of both a rotation and a translation. In this connection the following general statement is noteworthy: Let AB be the initial position, C being the middle point. Draw any curve to which AB is tangent at C and such that, as one passes along the curve from C, the angle between its tangent and the initial direction never diminishes. Then, in order that AB shall sweep over a minimum area in changing its direction by a given amount, θ, it suffices to slide it along this curve in such a way as to be always tangent to it at the mid-point C, the motion to continue until a final position $A'B'$ has been reached whose direction makes the angle θ with the initial direction. In order to see this, we need only note that for any such movement of a line-segment the instantaneous center always lies on the perpendicular bisector.*

4. *The Original Problem.* Returning to the original problem of § 1, we see that it concerns a rotation of the segment through the special angle 180° with the further restriction that it shall finally return completely upon itself. In order for this to be accomplished by a movement which shall at all times have its instantaneous center upon the perpendicular bisector of the segment and hence, in accordance with § 3, shall sweep *over* a minimum area, it evidently suffices to give the segment a simple rotation of 180° about its own middle point, regarded as fixed. This, however, is not the only

* As noted earlier, we are supposing, as the problem implies, that during the motion the direction of the segment changes only monotonically. It is for this reason that the single condition stated above concerning the *shape* of the curve is necessary.

possible solution, though it is the simplest. There are, in fact, an infinite number of other ways of producing the desired result. For example, first give the segment a pure translation by sliding it lengthwise any given distance along the indefinitely long line of which it forms a part, then give it a pure rotation of 180° about its middle point, then slide it back along the same line as before until it takes the desired position upon itself. For all such methods the instantaneous center always lies on the perpendicular bisector, this point being at infinity in the case of movements of pure translation.

5. *Problem of* § *3.* We now proceed to consider the problem of § 3, and eventually that of § 1, when area swept *out* instead of swept *over* is to be minimized. The simplest case which can then arise is that in which, during the movement, no area is passed over more than twice; the next simplest case is that in which no area is passed over more than three times; next the case in which none is passed over more than four times; etc. Let us take for the moment the most general case; namely, that in which a certain area is passed over n times, but none more than this number of times. If, then, we let T represent the area swept *over* as the segment changes its direction by the amount θ, and let S_1 be the area passed over twice (duplicated), S_2 the area passed over three times (triplicated), $\cdots$, S_{n-1} the area passed over n times, we shall have as an equation for determining the area Σ swept *out:*

$$(1) \qquad \Sigma = T - S_1 - 2S_2 - 3S_3 - \cdots - (n-1)S_{n-1}.$$

We now proceed to consider in detail the simplest case; namely, that in which only duplication is present. We have $S_2 = S_3 = \cdots = S_{n-1} = 0$, so that (1) reduces to

$$(2) \qquad \Sigma = T - S_1.$$

Moreover, the greatest value which S_1 can take is $\frac{1}{2}T$, this corresponding to the extreme assumption that the entire area swept over is duplicated, as takes place for example when a segment is rotated through 360° about a fixed point in its perpendicular bisector. Thus, in addition to (2), we have

$$(3) \qquad S_1 \leqq \tfrac{1}{2}T.$$

It follows that any such movement must belong to one of the following four classes: (*a*) T not a minimum and $S_1 < \frac{1}{2}T$, (*b*) T not a minimum, but $S_1 = \frac{1}{2}T$, (*c*) T a minimum, but $S_1 < \frac{1}{2}T$, (*d*) T a minimum and $S_1 = \frac{1}{2}T$.

Of these it is easy to show that Σ can be a minimum only in case (*d*), it being assumed for the moment that there is a geometric possibility of a movement (or movements) in which (*d*) is realized. In fact, recalling from § 3 that when T is a minimum it has the value $l^2\theta$, we may write, corresponding to the four cases, the following:

$$
\begin{aligned}
&(a)\ \Sigma = T - S_1 > T - \tfrac{1}{2}T = \tfrac{1}{2}T > \tfrac{1}{2}l^2\theta,\\
&(b)\ \Sigma = T - S_1 = T - \tfrac{1}{2}T = \tfrac{1}{2}T > \tfrac{1}{2}l^2\theta,\\
&(c)\ \Sigma = T - S_1 > T - \tfrac{1}{2}T = \tfrac{1}{2}T = \tfrac{1}{2}l^2\theta,\\
&(d)\ \Sigma = T - S_1 = T - \tfrac{1}{2}T = \tfrac{1}{2}T = \tfrac{1}{2}l^2\theta.
\end{aligned}
$$

Thus, it is possible for Σ to attain its smallest value only in case (*d*); that is, T itself must be a minimum and $S_1 = \frac{1}{2}T$. Moreover, this smallest value of Σ (if geometrically realizable) has the value $\frac{1}{2}l^2\theta$, which we shall hereafter refer to as the *absolute minimum* for duplication.

It only remains to consider whether movements of class (*d*) are actually possible, and for this let us refer again to Fig. 1. Here, as the segment is rotated from AB to $A'B'$ the area S is duplicated, while R and T are passed over but once. However, by taking the radius OC sufficiently large, the values of R and T may be brought as near to zero as we please, thus leaving only the duplicated area S. Since, by § 3, T is a minimum for all such movements, it appears that the conditions represented in case (*d*), while not actually realizable in Fig. 1, may be made as nearly so as we please by choosing a sufficiently large radius OC. By such a movement, therefore, the value of Σ may be brought as near as we please to its absolute minimum, $\frac{1}{2}l^2\theta$, though not to this actual value. Moreover, it appears from geometrical considerations that no other method of moving the segment will produce similar conditions, for if the sliding of the segment is not done along a circular arc (the mid-point C always remaining the point of tangency in order to have T a minimum) then not all the area corresponding to S will be duplicated. In fact, the area generated by CB will then be either too large or too small to be exactly covered later by the area generated by end CA in its forward movement.

We therefore conclude that the problem of § 3, when considered with reference to area swept *out*, and with the assumption that no area is passed over by the moving segment more than twice, has no solution; that is, there is no one method of movement that sweeps out a minimum area. Nevertheless, the area in question may be brought as near as we please to the value $\frac{1}{2}l^2\theta$, the absolute, though unattainable, minimum.

Before passing to the consideration of similar questions when triplication is allowable, it is of interest to apply the results just noted to the special problem of § 1. Here $\theta = 180°$ and we have the further condition that the segment is finally to rest upon itself. The area swept over in doing this may be brought as near as we please to the value $\frac{1}{2}\pi l^2$, thus conforming to the above general statement, as follows: Let AB (Fig. 1) be the initial position and choose $A'B'$ so that θ shall be arbitrarily near to 180° (though not equal to this amount). Moreover, suppose that AB and $A'B'$ as thus drawn are extended until they meet, say at the point P. Consider now the movement obtained by first rotating the segment in the manner indicated in Fig. 1 to the position $A'B'$, then sliding it lengthwise along $B'A'$ produced until the midpoint lies at P, then a rotation about P until the segment lies lengthwise upon AB produced, and finally a sliding back along AB thus produced until the segment lies completely upon its original position. Evidently, in accordance with the general results above obtained, the area thus swept over can, by choosing θ sufficiently near to 180° and OC sufficiently large, be brought arbitrarily near to the indicated amount, $\frac{1}{2}\pi l^2$.

6. *Triplication of Areas.* We pass on to analogous studies when triplication as well as duplication is allowed. Here, instead of (2) and (3), we have respectively

$$(4) \qquad \Sigma = T - S_1 - 2S_2,$$

$$(5) \qquad S_2 \leqq \tfrac{1}{3}T,$$

the equality sign in (5) corresponding to the extreme assumption that the entire area swept over is triplicated.

From (4) we have $\Sigma + S_1 + S_2 = T - S_2$. But $S_1 + S_2 \leqq \Sigma$. Hence, $2\Sigma \geqq T - S_2$, or

$$(6) \qquad \Sigma \geqq \tfrac{1}{2}(T - S_2).$$

Corresponding to the four cases (a), (b), (c), (d) of § 5, we may now consider the following as representing all possibilities: (a) T not a minimum and $S_2 < \frac{1}{3}T$, (b) T not a minimum, but $S_2 = \frac{1}{3}T$, (c) T a minimum, but $S_2 < \frac{1}{3}T$, (d) T a minimum and $S_2 = \frac{1}{3}T$. These assumptions, when employed in (6), lead to the following results respectively:

$$(a) \quad \Sigma \geqq \tfrac{1}{2}(T - S_2) > \tfrac{1}{2}(T - \tfrac{1}{3}T) = \tfrac{1}{3}T > \tfrac{1}{3}l^2\theta,$$

$$(b) \quad \Sigma \geqq \tfrac{1}{2}(T - S_2) = \tfrac{1}{2}(T - \tfrac{1}{3}T) = \tfrac{1}{3}T > \tfrac{1}{3}l^2\theta,$$

$$(c) \quad \Sigma \geqq \tfrac{1}{2}(T - S_2) > \tfrac{1}{2}(T - \tfrac{1}{3}T) = \tfrac{1}{3}T = \tfrac{1}{3}l^2\theta,$$

$$(d) \quad \Sigma \geqq \tfrac{1}{2}(T - S_2) = \tfrac{1}{2}(T - \tfrac{1}{3}T) = \tfrac{1}{3}T = \tfrac{1}{3}l^2\theta.$$

Thus, it is only in case (d) that Σ can attain as low a value as $\frac{1}{3}l^2\theta$, which value, corresponding to the procedure of § 5, we may now take as the absolute minimum for triplication.

The geometric interpretation of these results, however, presents more serious difficulties than in the analogous results for duplication, for, if T be a minimum, as (d) requires, it is not apparent that triplication can be present in any species of movement to such an extent as to cover the entire area swept over, as (d) likewise requires, nor does it appear that such conditions can be realized through any limiting form of movement analogous to that presented in the expanding circular ring already employed in the case of duplication. It may therefore well be (and it is the author's belief, though he cannot furnish a formal proof of the fact) that the value $\frac{1}{2}l^2\theta$, already met with as the lowest approachable value in the case of duplication, is likewise the lowest approachable value even when triplication is allowed, the only difference in the two cases being that this value may actually be attained in the latter case for one or more special values of θ.*

Finally, it may be observed that the values of the so-called absolute minima for the cases where area may be passed over four, five, six, $\cdots$ times are respectively $\frac{1}{4}l^2\theta$, $\frac{1}{5}l^2\theta$, $\frac{1}{6}l^2\theta$, $\cdots$. The consideration of these cases, however, on the geometrical side again presents serious difficulties, but tends to the opinion, as in the case of triplication, that *in general* the smallest area that can be swept over by any actual movement of angle θ is $\frac{1}{2}l^2\theta$ rather than any of these smaller values.

UNIVERSITY OF MICHIGAN.

CONVERGENCE OF SEQUENCES OF LINEAR OPERATIONS †

BY T. H. HILDEBRANDT.

Let U_n be a sequence of linear continuous operations on the class F of functions f, continuous on the interval (a, b), i.e., suppose that every U satisfies the two conditions:

$$U(c_1f_1 + c_2f_2) = c_1U(f_1) + c_2U(f_2) \tag{1}$$

for every pair of constants (c_1, c_2) and every pair of functions (f_1, f_2) of the class F;

(2) There exists a constant M depending on U such that if Nf is the maximum value of $|f|$ on (a, b) then

$$|U(f)| \leqq MNf.$$

The greatest lower bound of all possible values M might be called the *modulus* of U.

* Thus, in case $\theta = \pi$ and triplication is allowed, the corresponding value $\frac{1}{2}l^2\pi$ may be attained as follows: Construct the hypocycloid of three cusps obtained by rolling the circle of radius $\frac{1}{2}l$ within the circle of radius $\frac{3}{2}l$ and let the given segment (of length $2l$) move so as to be always tangent to this curve and yet be everywhere entirely within it. The resulting area swept over as θ passes from 0 to π is entirely triplicated, as is well known, and is equal to the amount above stated, $\frac{1}{2}l^2\pi$. See, for example, F. Gomes Teixeira, *Traité des Courbes Spéciales Remarquables Planes et Gauches*, vol. II, p. 193. (Coïmbre, 1909.)

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