THE TRANSFORMATION OF ELLIPTIC INTEGRALS.

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1. *Introduction.* Jacobi discovered the transformation of the fifth order and proposed the problem of the transformation of order n. The solution of a system of algebraic equations is required. The number of arbitrary quantities is equal to the number of equations, but the direct solution could not be effected. The introduction of the inverse functions and the periods furnished a transcendental solution which cannot be regarded as complete till the transcendental elements are eliminated since the periods are not given. Cayley attempted an algebraic solution without success, as did also Clifford. Cayley says:

"The extension of this algebraic theory (Jacobi's determination of the transformations of degrees 3 and 5) to any value whatever of n is a problem of great interest and difficulty: such theory should admit of being treated in a purely algebraical manner; but the difficulties are so great that it was found necessary to discuss it by means of the formulae of the transcendental theory, in particular by means of the expressions involving Jacobi's q (the exponential of $-\pi k'/k$), $\cdots$. In the present memoir I carry on the theory algebraically as far as I am able; and I have, it appears to me, put the purely algebraical question in a clearer light than has hitherto been done; but I still find it necessary to resort to the transcendental theory."

In what follows the solution of Jacobi's system of equations is given independently of the transcendental theory and the foundation laid for a purely algebraic treatment of the whole subject of transformations.

2. *Jacobi's Problem.* Let s, Σ be two conics such that it is possible to find a polygon of n sides inscribed in s and circumscribed about Σ. Then, by a well known theorem, there can be described an infinity of polygons having the same property. Let a parameter t be introduced on the conic s; then the values of the parameters of the vertices of any polygon of the system are given by an equation $f(t, \lambda) = 0$ of degree n in t and of the first degree in λ.

To see this, let the equation of s be $y^2 + xz = 0$ or $x = t^2$, $y = t$, $z = 1$. Then any quadratic equation $at^2 + bt + c \equiv a(t - t_1)(t - t_2) = 0$ may be regarded as determining the line joining the points whose parameters are t_1 and t_2. Any equation $f(t_1, t_2) = 0$ may be regarded as defining an envelope, viz. of those lines determined by values t_1, t_2 satisfying the equation. If the equation is of degree n and symmetric in t_1 and t_2 the curve enveloped is of class n, if unsymmetric of class $n + m$ where n and m are the degrees in t_1, t_2. Let $A_0, \cdots, A_n$ represent $n + 1$ points on s; then the equation $\alpha_0/A_0 + \cdots + \alpha_n/A_n$ represents for different values of α_i curves tangent to the lines joining any two of the points A_i. This equation may be written in the form

$$\Sigma \frac{\alpha_i}{(t_1 - \alpha_i)(t_2 - \alpha_i)} = 0, \qquad \text{or} \qquad \Sigma \frac{\alpha_i}{t_1 - a_i} = \Sigma \frac{\alpha_i}{t_2 - a_i},$$

or

$$\frac{f(t_1) + \lambda\varphi(t_1)}{\varphi(t_1)} = \frac{f(t_2) + \lambda\varphi(t_2)}{\varphi(t_2)}.$$

This curve is tangent to the lines joining pairs of points given by $f(t) + \lambda\varphi(t) = 0$ and is of class n. It may be seen also that when a curve of class n touches all lines joining pairs of a system of $n + 1$ points on a conic it touches the lines formed from an infinity of systems of $n + 1$ points on the conic.

Suppose a conic Σ tangent to $n + 1$ of the lines, the constants α_i can be determined so that the curve of equation $\Sigma\, \alpha_i/A_i = 0$ touches n additional tangents of Σ, or so that the curve and Σ have $2n + 1$ tangents in common, or so that the curve must decompose and contain Σ as one part together with a residue Γ. The curve $\Sigma\Gamma$ is tangent to the lines joining pairs of a succession of systems of $n + 1$ points on the conic s and the conic Σ is inscribed in a polygon of $n + 1$ sides connecting the points of each system. The curve Γ also decomposes into conics or conics and a point when $n + 1$ is odd. For let $t_1, \cdots, t_{n+1}$ be the parameters of the vertices of any polygon circumscribed about Σ and belonging to the systems and suppose them taken in order. Then the line joining t_i to t_{i+p} envelops a conic. For the relation between t_i and t_{i+p} is doubly quadratic, since to t_i correspond t_{i+p} and t_{i-p} (the subscripts taken modulo $n + 1$). Hence this relation must be of the form

$$A(t_1^2 + t_2^2) + Bt_1t_2 + Ct_1t_2(t_1 + t_2) + D(t_1 + t_2) + Et_1^2t_2^2 + F = 0,$$

being necessarily symmetric. The envelope of the line must be of the second class. If $t_{i+p} = t_{i-p}$, the relation must be doubly linear and the envelope must be of the first class. This would occur if $n + 1 = 2m$ and $p = m$.

The polynomial whose roots are $t_1, \cdots, t_{n+1}$, as has been seen, must be of the form $f(t) + \lambda\varphi(t)$, or must belong to an involution. The double elements of the involution are $2n$ in number and are the parameters of points in which two vertices of a polygon coincide. These polygons are found by starting from a common point of s and Σ or from a point of contact of a common tangent. If $n + 1$ is odd, starting from a common point the polygon must consist of a succession of segments counted twice, and the tangent at the extremity, which must be a tangent of Σ. There are four such polygons, and the corresponding forms of the involution must be

$$f + \lambda_i\varphi = (t - \alpha_i)\psi_i^2 \qquad (i = 1, 2, 3, 4)$$

where α_i is the parameter of an intersection. If $n + 1$ is even, the singular polygons connect intersections or connect contact points of common tangents, and the forms of the involution are

$$f + \lambda_1\varphi = (t - \alpha_1)(t - \alpha_2)\psi_1^2, \; f + \lambda_3\varphi = (t - \beta_1^2)(t - \beta_2^2)\psi_3^2,$$

$$f + \lambda_2\varphi = (t - \alpha_3)(t - \alpha_4)\psi_2^2, \; f + \lambda_4\varphi = (t - \beta_3^2)(t - \beta_4)^2\psi_4^2,$$

where β_1, β_2, β_3 and β_4 are the parameters of the points of contact. Since the double elements are properly accounted for, there are only four branch forms.

The forms f, φ are seen to be subject to the same conditions as are the forms in Jacobi's problem. Conversely, if the involution $f + \lambda\varphi$ possesses branch forms of the same character as those that are required in Jacobi's problem, the forms f and φ lead to two conics in the poristic relation, and the involution curve decomposes. Suppose $n + 1$ odd and equal to $2m + 1$. Then to a polygon $(t - \alpha)\psi^2$ correspond $m(m - 1)/2$ double tangents of the involution curve, or $2m(m - 1)$ for the four forms. A curve of class $2m$ can only have this number if it decomposes. Every addition to the maximum number of double tangents for a proper curve must be due to a further decomposition. For we have

$$1 + \frac{(p+q-1)(p+q-2)}{1}$$
$$= \frac{(p-1)(p-2)}{2} + \frac{(q-1)(q-2)}{2} + pq,$$

and since

$$2m(m-1) = \frac{(2m-1)(2m-2)}{2} + m - 1,$$

the involution curve must consist of m parts. In the special case where the curve is an involution curve these must all be conics. For let $p_1, \cdots, p_m$ be the classes of the components, so that $p_1 + \cdots + p_m = 2m$; then some class must be equal to 2 or to 1. But if a point p forms part of the involution curve, the order of the involution is even, since points on s collinear with P belong simultaneously to the involution. The class of some part must be 2 and a conic Σ must be inscribed in lines of the system. If Σ is tangent to fewer than $n+1$ lines, the forms f and φ must have a common factor, which is excluded by the assumption of a proper solution of Jacobi's problem. Then Σ touches $n+1$ lines, and by the theory given above the involution curve completely decomposes into conics. If $n+1$ is even the conclusion is similar: the involution curve consists of conics and one point.

3. *Closure.* To effect the solution of Jacobi's system, it is necessary to consider the condition for closure. This is known under various forms. It is convenient to use a recurrence formula. Let there be two conics referred to the common self-polar triangle (A) $x^2 + y^2 + z^2 = 0$, (B) $ax^2 + by^2 + cz^2 = 0$. Take M_0 a point on A and let D_1, the polar of M_0 with respect to B, meet A in M_1, M_{-1}. The polar of M_1 meets A in M_0 and a point M_2 and the polar of M_{-1} meets A in M_0 and M_{-2}. Let D_2 be the line M_2M_{-2}. In this way may be derived a series of points M_pM_{-p} and lines D_p; the envelope of D_p may be called A_p. From these definitions it follows, letting M_0 be $\xi\eta\zeta$, that the equation of D_0 is $\xi x + \eta y + \zeta \tau = 0$, that of D_1 is $a\xi x + b\eta y + c\zeta z = 0$, and the equation of D_n is $a_n\xi x + b_n\eta y + c_n\zeta z = 0$, where a_n, b_n, and c_n are functions of a, b, and c independent of $\xi\eta\zeta$ and determined by the following relations of recurrence:

$$a_0 = b_0 = c_0 = 1, \qquad a_1 = a, \qquad b_1 = b, \qquad c_1 = c,$$
$$b_{2p} + c_{2p} = 2b_p^2 c_p^2, \qquad cb_{2p-1} + bc_{2p-1} = 2b_{p-1}c_{p-1}b_p c_p,$$
$$c_{2p} + a_{2p} = 2c_p^2 a_p^2, \qquad ac_{2p-1} + ca_{2p-1} = 2c_{p-1}a_{p-1}c_p a_p,$$
$$a_{2p} + b_{2p} = 2a_p^2 b_p^2, \qquad ba_{2p-1} + ab_{2p-1} = 2a_{p-1}b_{p-1}a_p b_p.$$

There results the system of equations

$$\frac{b_n^2 - c_n^2}{b^2 - c^2} = \frac{c_n^2 - a_n^2}{c^2 - a^2} = \frac{a_n^2 - b_n^2}{a^2 - b^2},$$
$$\frac{b^2 c_n^2 - c^2 b_n^2}{b^2 - c^2} = \frac{c^2 a_n^2 - a^2 c_n^2}{c^2 - a^2} = \frac{a^2 b_n^2 - b^2 a_n^2}{a^2 - b^2},$$

from which follow

$$a_n^2 = G_n a^2 + H_n, \qquad b_n^2 = G_n b^2 + H_n, \qquad c_n^2 = G_n c^2 + H_n,$$

where G_n and H_n are symmetric functions of a^2, b^2, and c^2, with $G_0 = 0$, $H_0 = 1$, $G_1 = 1$, $H_1 = 0$. They satisfy the equations

$$G_{p-1}G_{p+1} = H_p^2, \qquad G_{2p+1} = (G_p H_{p+1} - G_{p+1}H_p)^2,$$
$$G_{2p} = 4a_p^2 b_p^2 c_p^2 G_p, \qquad G_2 = 4a^2b^2c^2.$$

Hence G_n is the square of a symmetric function of degree $n^2 - 1$ in a, b, and c; i.e. $G_n = \Lambda_n^2$. The sign of Λ_n is determined by the equations $\Lambda_2 = 2abc$, $\Lambda_{p-1}\Lambda_{p+1} = -H_p$.

The envelope of D_n is

$$a_n^2 x^2 + b_n^2 y^2 + c_n^2 z^2,$$

or

$$\Lambda_n^2(a^2x^2 + b^2y^2 + c^2z^2) - \Lambda_{n-1}\Lambda_{n+1}(x^2 + y^2 + z^2) = 0.$$

If A_n coincides with A_0 the condition is $\Lambda_n = 0$. The condition that a polygon of n sides inscribed in the conic $x^2 + y^2 + z^2 = 0$ is circumscribed about $a^2x^2 + b^2y^2 + c^2z^2 = 0$ is $\Lambda_n = 0$.

Involutions of the required character may be constructed from the equations of the lines D_p. Taking the cubic involution, we find that the condition for triangular closure is

$$(bc+ca+ab)(-bc+ca+ab)(bc-ca+ab)(bc+ca-ab)=0$$

or

$$b_2c_2 + c_2a_2 + a_2b_2 = 0.$$

If we set $\xi = 2t_0$, $\eta = 1 - t_0^2$, $\zeta = i(1 + t_0^2)$, the equation $f = 0$, where

$$f = (t - t_0)\{4a_2t_0t + b_2(1 - t_0^2)(1 - t^2) - c_2(1 + t_0^2)(1 + t^2)\},$$

gives the parameters of the points M_0, M_2, and M_{-2}. If we suppose $b_2c_2 + c_2a_2 + a_2b_2 = 0$, and let t_0 vary, the forms of a cubic involution are determined. In fact, the factor

$$4a_2t_0t + b_2(1 - t_0^2)(1 - t^2) - c_2(1 + t_0^2)(1 + t^2)$$

gives the affinity equation of the involution when equated to zero. The double elements are found by putting $t_0 = t$ and they are the roots of the equation

$$4a_2t^2 + b_2(1 - t^2)^2 - c_2(1 + t^2)^2 = 0.$$

Let us call these roots δ_1, δ_2, δ_3, and δ_4. The branch elements ϵ_1, ϵ_2, ϵ_3, and ϵ_4 are the values of t_0 such that the affinity quadratic is a square. Hence they must satisfy the equation

$$(c_2^2 - b_2^2)t^4 + 2(a_2^2 + b_2^2 - 2a_2^2)t^2 + c_2^2 - b_2^2 = 0.$$

The equations for δ and ϵ are reciprocal equations involving only even powers of t, and it may easily be found that the branch forms of the involution are of the form

$$\lambda f + \mu\varphi = A_1(x + \epsilon)(x + \delta)^2, \qquad \mu f + \lambda\varphi = A_3\left(x + \frac{1}{\epsilon}\right)\left(x + \frac{1}{\delta}\right)^2,$$

$$\lambda f - \mu\varphi = A_2(x - \epsilon)(x - \delta)^2, \qquad \mu f + \lambda\varphi = A_4\left(x - \frac{1}{\epsilon}\right)\left(x - \frac{1}{\delta}\right)^2,$$

where δ is a root of the equation for the double elements and ϵ is the corresponding value of the branch element. It is easy to complete the solution of the transformation problem. It is seen that Cayley's normal form of the elliptic integral appears here.

If the order is 4, the involution is given by the equation

$$f = \{4at_0t + b(1 - t_0^2)(1 - t^2) - c(1 + t_0^2)(1 + t^2)\}\{4a_3t_0t + b_3(1 - t_0^2)(1 + t^2) - c_3(1 + t_0^2)(1 + t^2)\},$$

with $\Lambda_4 = 4abca_2b_2c_2 = 0$, the roots of $f = 0$ being the parameters of the points M_1, M_{-1}, M_3, and M_{-3}. Suppose $a_2 = 0$, then $a_3 = -ab_2c_2$, $b_3 = bb_2c_2$, and $c_3 = cb_2c_2$; hence, omitting the factor b_2c_2, the involution consists of the forms

$$b^2(1-t_0^2)^2(1-t^2)^2+c^2(1+t_0^2)^2(1+t^2)^2 - 2bc(1-t_0^4)(1-t^4)-16a^2t_0^2t^2,$$

with $a_2 = -b^2c^2+c^2a^2+a^2b^2=0$, or

$$[(c-b)t_0^2+c+b]^2t^4+[(c+b)t_0^2+c-b]^2 + 2\frac{c^2-b^2}{c^2+b^2}[(c^2+b^2)t_0^4+2(c^2-b^2)t_0^2+c^2+b^2]t^2,$$

or, if we set $[(c-b)t_0^2+c+b]^2=\lambda$ and $[(c+b)t_0^2+c-b]^2=\mu$, of the forms

$$\lambda t^4+\frac{c^2-b^2}{c^2+b^2}(\lambda+\mu)t^2+\mu.$$

For the fifth order the affinity equation is

$$[4a_2t_0t+b_2(1-t_0^2)(1-t^2)-c_2(1+t_0^2)(1+t^2)] \times [4a_4t_0t-b_4(1-t_0^2)(1-t^2)-c_4(1+t_0^2)(1+t^2)]$$

with $\Lambda_5=0$. If we set $p=b_2c_2$, $q=c_2a_2$ and $r=a_2b_2$ it is found that

$$pqr\cdot\Lambda_5=(pq+qr-rp)(qr+rp-pq)(rp+pq-qr) - pqr(p+q+r)^3$$

and

$$a_4=q^2+r^2-p^2,\quad b_4=r^2+p^2-q^2,\quad c_4=p^2+q^2-r^2.$$

The discriminant of the involution is found by putting $t_0=t$ in the affinity equation. It must also be given by the resultant of the two factors, since a, b, and c are subject to the relation $\Lambda_5=0$. If we develop the two forms of the discriminant and compare the coefficients, a number of forms of the closure condition result, but there are extraneous factors.

This method of constructing involutions is general and furnishes a complete solution of Jacobi's problem. The problem is always possible and determinate. It has been assumed that a, b, and c are subject to no other relation than $\Lambda_n=0$. Such cases are of importance in the theory. Since the present purpose is to solve Jacobi's problem in its general form, they are left for further consideration.

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BACHMANN ON FERMAT'S LAST THEOREM.

Das Fermatproblem in seiner bisherigen Entwickelung. By Paul Bachmann. Berlin and Leipzig, Walter de Gruyter, 1919. pp. viii + 160.

This volume reproduces to a considerable extent most of the important contributions which have so far been made toward a proof of Fermat's last theorem. It is far more complete than anything of the sort heretofore published. In particular, a reader of the book will find therein an account of the main results of Kummer, with proofs in most cases set forth in full. The writer wishes to call attention to the fact, however, that a number of references to articles bearing directly on some of the work given in the text have been omitted by Bachmann, a few of which will be noted, in detail, presently. If a better historical perspective is desired, it would be well for a reader to examine at the same time chapter 26, volume 2, of Dickson's *History of the Theory of Numbers.*

I shall now point out some parts of the text which give an account of results not given in detail elsewhere, aside from the original articles.* Consider

$$x^p + y^p + z^p = 0, \tag{1}$$

where x, y and z are rational integers, prime to each other, and p is an odd prime. The assumption that xyz is prime to p

* For an account of the more elementary results regarding the theorem, cf. Carmichael, *Diophantine Analysis*, chap. 5, or Bachmann, *Niedere Zahlentheorie*, vol. 2, chap. 9.