ON THE FOURIER COEFFICIENTS OF A CONTINUOUS FUNCTION.

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It is well known that when

$$\frac{a_0}{2}+\sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta)$$

is the Fourier expansion of a function $f(\theta)$ which is real and continuous for $0 \leqq \theta \leqq 2\pi$, then $\Sigma(a_n^2 + b_n^2)$ converges. Here the exponent 2 cannot in general be replaced by a smaller one; in fact, Carleman* has constructed an example of a continuous $f(\theta)$ where $\Sigma(a_n^{2-2\delta} + b_n^{2-2\delta})$ diverges for any $\delta > 0$, and this example has been simplified by Landau.†

In the present note it will be shown that, *given any single-valued real function $\varphi(x)$, subject only to the condition that $\varphi(x)$ becomes infinite as x becomes infinite, there exists a real continuous function $f(\theta)$ whose Fourier coefficients a_n, b_n make the series*

$$\sum (a_n^2 + b_n^2)\varphi\left(\frac{1}{a_n^2 + b_n^2}\right)$$

divergent. Assuming $\varphi(x) = x^\delta$, where $\delta > 0$, and observing that $(a^2 + b^2)^{1-\delta} < a^{2-2\delta} + b^{2-2\delta}$, we have the particular result referred to above.

If we denote by $f_1(\theta)$ the function conjugate to $f(\theta)$, and write $z = e^{\theta i}$, $F(z) = f(\theta) + if_1(\theta)$, the Fourier expansion of $F(z)$ is $\Sigma_0^\infty c_n z^n$, where $c_0 = a_0/2$, $c_n = a_n - ib_n$ $(n > 0)$. Our statement will be proved by constructing a function $F(z)$ continuous for $|z| = 1$ and such that $\Sigma |c_n|^2\varphi(1/|c_n|^2)$ diverges. This will be done by means of the following result due to Hardy and Littlewood‡ and used by Landau, loc. cit., for a different purpose:

* T. Carleman, *Ueber die Fourierkoeffizienten einer stetigen Funktion*, ACTA MATH., vol. 41 (1918), pp. 377–384.

† E. Landau, *Bemerkungen zu einer Arbeit des Herrn Carleman*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 147–153.

‡ G. H. Hardy and J. E. Littlewood, *Some problems of diophantine approximation*, ACTA MATH., vol. 37 (1914), pp. 155–239. See p. 220.

Let ξ be a real irrational number such that all the denominators in its expansion in a continued fraction are bounded (for instance $\xi = \sqrt{2}$ or any quadratic irrationality). Then there exists an $A = A(\xi)$ independent of n and z such that for any $n \geqq 1$, and any z on the unit circle $|z| = 1$,

$$\left|\sum_{\nu=1}^{n} e^{\nu^2\pi\xi i} z^\nu\right| < A\sqrt{n}.$$

Making

$$F_\nu(z) = \sum_{\mu=1}^{n_\nu} \frac{e^{\mu^2\pi\xi i}}{\sqrt{n_\nu}} z^\mu,$$

we have therefore $|F_\nu(z)| < A$ for $|z| = 1$; writing $k_\nu = n_0 + n_1 + \cdots + n_{\nu-1}$ and assuming d_ν to be such that $\Sigma|d_\nu|$ converges, we find that the series

$$F(z) = \sum_{\nu=0}^{\infty} d_\nu z^{k_\nu} F_\nu(z)$$

converges uniformly for $|z| = 1$, so that $F(z)$ is continuous on the unit circle. Multiplying by $z^{-n-1}\,dz$ and integrating along the unit circle, we may integrate term by term to the right on account of the uniform convergence, and the Fourier coefficients c_n of $F(z)$ are thus found to be

$$c_n = d_\nu \frac{e^{\mu^2\pi\xi i}}{\sqrt{n_\nu}} \qquad (n = k_\nu + 1,\ k_\nu + 2,\ \cdots,\ k_\nu + n_\nu).$$

Consequently

$$\sum_{n=k_\nu+1}^{k_{\nu+1}} |c_n|^2 \varphi\left(\frac{1}{|c_n|^2}\right) = |d_\nu|^2 \varphi\left(\frac{n_\nu}{|d_\nu|^2}\right),$$

and since $\varphi(x)$ becomes infinite as x becomes infinite, we may choose each n_ν so that

$$\varphi\left(\frac{n_\nu}{|d_\nu|^2}\right) > \frac{|D_\nu|}{|d_\nu|^2},$$

where $\Sigma|D_\nu|$ is any given divergent series. With this choice of n_ν, it follows that $\Sigma|c_n|^2\varphi(1/|c_n|^2)$ diverges, which proves our theorem.

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