NOTES ON ELECTRICAL THEORY.

BY PROFESSOR H. BATEMAN.

1. *The Production of Light.*—According to an idea that is generally associated with the names of Faraday and J. J. Thomson, energy is radiated from an electric charge in the form of light or electric waves when the lines of electric force issuing from the charge assume a wavy form. This idea has been made more definite by a study of the form of the lines of force in various types of electromagnetic fields. It is known now that waves on the lines of force may be produced either by an oscillation of the electric charge or by the continual emission of either electric or magnetic doublets. The emission of a single electric charge produces a kink in each line of force while the emission of a single magnetic charge produces a rotation of each line of force. When charges of opposite signs are successively emitted in the same direction so as to be equivalent to a doublet, the successive kinks are in opposite directions and are thus equivalent to a solitary wave. When a succession of magnetic poles of one sign are emitted in one direction, the lines of force rotate about this direction but they rotate with different angular velocities. The same is true when magnetic poles of opposite signs are fired out in opposite directions.

If a magnetic pole emitted in one direction is followed by a magnetic pole of the opposite sign so as to give a magnetic doublet, the lines of force rotate first in one direction and then in the other returning finally to their original directions. The result again is that a solitary wave runs along each line of force, but just as in the electrical case one line of force is unaffected.

The question now arises whether two types of light production really occur in nature. This is a question that has been bothering physicists for some time and opinion is divided. An argument in favor of the dual nature of light may be presented as follows:

2. *The Reaction of an Electric Charge upon a Field which Alters its Motion.*—It is usually assumed in electromagnetic theory that when an electric charge A is accelerated by an

external electromagnetic field E, the field E is modified indirectly only by the action of the electron's field on the sources of the field E. This reaction is, however, an after effect and it seems more in accordance with the principles of mechanics to suppose that the acceleration of A is accompanied by a direct local modification of the field E. Of course the acceleration of A produces a modification of A's field and it may be thought that this is the only local modification that occurs, but this modification spreads out in all directions while A's motion is changed only in one direction. At any rate the hypothesis that the field E is locally modified is worth considering; it may be that the disregard of this modification has been the cause of the present difficulties in the electromagnetic theory of radiation.

To get an idea of the nature of this local modification we shall adopt a definite hypothesis with regard to the structure of a line of electric force as viewed by an imaginary observer who is able to distinguish particles that are extremely small and exceedingly close together.

We shall suppose that a line of electric force issuing from a stationary positive charge P is built up of a series of doublets arranged as follows:

$$P + \quad -+ \quad -+ \quad -+ \quad -+ \quad A \quad -+ \quad -+ \quad -+ \quad -+ \quad \rightarrow$$

These doublets may be supposed to be moving in the direction of the arrow with the velocity of light. The free charge at P is only temporarily at rest, for it may be supposed to secure a negatively charged partner from a doublet which arrives at P and move away with the velocity of light, leaving the positive constituent of the doublet at P until this charge in its turn secures a negative partner. The charge at P may be really moving all the time along a zig-zag path in the neighborhood of the point P, the mean free path being comparable in size with the so-called "diameter" of the positive charge. If the mass of the charge is proportional to the number of collisions with doublets per unit time it is easy to understand why the mass is inversely proportional to the "diameter" or mean free path of the free electric charge, especially if the free charge always moves with the velocity of light in describing its zig-zag path. If n denotes the number of collisions per second and m the mass of the positive charge, a lower limit to the value of n may, perhaps, be given by the equation $mc^2 = hn$, where c denotes the velocity of light and h, Planck's constant. The value of n for the nucleus of a hydrogen atom is then of order 10^{23}. So also if an electron is similarly constructed the number n is of order* 10^{20}. Let us now suppose that an electric charge of the same magnitude as one of the constituents of a doublet meets the line of electric force at A. If the charge is positive it may be supposed to seize the negative constituent of the doublet on the right leaving the positive constituent free. The effect is the same as if the free positive charge had been moved to the right, *i.e.* as if it had been repelled by the free positive charge at P. As for the newly formed doublet, it may be supposed to fly away in some direction which depends, perhaps, on the direction of motion of the free charge which entered at A. It is possible that the colliding charges are broken up into fragments which are scattered in all directions. This is a possibility that ought to be considered but at present it is simpler to assume that the charges remain intact.

In the latter case the result of the encounter between the free electric charge A and the line of force is that the free charge is displaced along the line of force, one of the doublets belonging to the line of force is annihilated and another doublet is formed and emitted in some direction which is probably different from that of the line of force. Mathematically the annihilation of one doublet and the emission of another may be represented by the emission of two doublets one of which travels in the direction of the doublet that was annihilated and is of such a nature as to completely annul the effect of this doublet. Where the former doublet would have a positive charge the emitted doublet must have an equal negative charge.

Electromagnetic fields in which pairs of doublets are emitted from a moving point have already been studied.† They may be derived from the elementary type of field in which charges which are equal but opposite in sign are emitted in different directions.

When the electric charge which meets the line of force at A

* If a free charge moves with the velocity of light, the mean length of the free path in the case of an electron may be of order 10^{-10}. This is about the order of magnitude of the radius of the ring electron. A free charge moving along a zig-zag path may be supposed to behave something like a ring electron.

† *Proc. London Math. Soc.*, (2), vol. 18 (1919), p. 95.

is negative, it may be supposed to seize the positive constituent of the doublet on the left, leaving the negative constituent behind. The result is equivalent to a displacement of the free charge towards the free positive charge at P just as if the negative charge were attracted by the positive charge. There is also a breaking up of a doublet belonging to the line of force and the emission of a new doublet in some direction. The displacement or acceleration of the free electric charge in the direction of the line of force may be regarded as an elementary process of which the equations of motion of an electric charge are a consequence. The free charge of an electron may be supposed to encounter at least 10^{10} lines of force from different charges in 10^{-10} seconds and the resultant acceleration may consequently be proportional to a mean value of the electric force in this interval of time. It is difficult to derive formally the equations of motion from our hypothesis regarding the structure of a line of force; the difficulty may arise from the fact that the true line of force of the familiar type of electromagnetic field is a limit of the discontinuous line of force which we have pictured in order to make the argument clear. The discontinuous structure may, however, be more nearly correct than the continuity with which we are so familiar. The idea of mass being proportional to a frequency of collision seems reasonable when compared with other physical laws such as the law of mass action in chemistry and Planck's relation between an energy quantum and frequency. The idea arose originally in connection with a theory of gravitation which has already been sketched elsewhere.*

3. *The Lines of Force of an Oscillating Electric Pole.*—It is known that a line of electric force of a moving electric pole can be considered as the locus of a series of light particles projected from the pole at successive instants.† If $\boldsymbol{v}$ denotes the velocity of the pole at time τ and the unit vector $\boldsymbol{s}$ indicates the direction of projection of the light particle emitted at this instant, $\boldsymbol{s}$ satisfies the differential equation

$$(c^2 - v^2)\boldsymbol{s}' = c\boldsymbol{s} \times \boldsymbol{v}' \times (c\boldsymbol{s} - \boldsymbol{v}),$$

where primes denote differentiations with respect to τ. When

* *Messenger of Mathematics*, vol. 48, Aug., 1918, p. 74.

† Leigh Page, *Amer. Journ. of Sci.*, vol. 38, Aug., 1914, p. 169; H. Bateman, this BULLETIN, March, 1915.

the motion is rectilinear and along the axis of z the above equation is easily solved. If (l, m, n) are the three components of $\boldsymbol{s}$, we have*

$$l = \frac{2\beta\sqrt{c^2 - v^2}\cos\alpha}{c + v + \beta^2(c - v)}, \qquad m = \frac{2\beta\sqrt{c^2 - v^2}\sin\alpha}{c + v + \beta^2(c - v)},$$

$$n = \frac{c + v - \beta^2(c - v)}{c + v + \beta^2(c - v)},$$

where α and β are constants for each line of force.

It is easy to see that n increases with v and is stationary when v is stationary. Let us consider the case of a simple periodic oscillation about the origin in which $v = ap\cos p\tau$. The extreme values of n are then given by $v = \pm ap$ and occur when the electric pole is at the origin. Let P be the position at time t of a light particle shot out in one extreme direction; then remembering that l', m', and n' are zero when the electric pole is at the origin, we see that as a point moves from P along the figures of the line of force at time t, the increments of the coordinates are

$$dx = -cld\tau, \qquad dy = -cmd\tau, \qquad dz = (v - cn)d\tau,$$

where $v = \pm ap$.

The direction of the tangent at P is found to be the same whether we take the upper or the lower sign in the expression for v. Let us draw a line OQ parallel to the tangent and study the relation of the line of force to this radial line OQ. The line of force evidently has a wavy form and P is the crest of one of the waves. The distance of P from the radius OQ is easily found to be

$$\frac{2\beta uc(t - \tau)}{[c^2(1 + \beta^2)^2 - u^2(1 - \beta^2)^2]^{1/2}},$$

where $u = \pm ap$. As we move from crest to crest going away from O, the height of a crest above OQ increases very slowly if u is small. Thus if we are considering the flight of a light particle, the height of a crest increases approximately at a rate

$$\frac{2\beta u}{1 + \beta^2},$$

* F. D. Murnaghan, *Amer. Journ. of Math.*, April, 1917; Johns Hopkins Circular, July, 1915.

which is equal to u when $\beta = \pm 1$ and is less than u for other values of β. When $\beta = 0$ the line of force is straight.

If the intensity of the ordinary light emitted from an oscillating electric pole depends on the square of the acceleration ap^2, then for a given intensity $u = ap$ is inversely proportional to p, so that for light of high frequency u may be very small indeed and the crests on a wavy line of force may be very close to the radial line OQ even at a great distance from the pole. It may be on this account that light of high frequency behaves much more nearly as if it were corpuscular in nature than light of low frequency.

The intensity of the light depends upon the slope of the line of force away from OQ rather than upon the distance of the crests from OQ. When v is very small in comparison with c the light vector at a great distance from O may be taken to be equal to $\boldsymbol{e}\boldsymbol{s}'/cr$, where e is the electric charge associated with the pole and r is the distance from the pole.

4. *Lines of Electric Force in Some Other Types of Electromagnetic Fields.*—If we suppose that an emission of light resulting from an acceleration of an electron is accompanied by an emission of doublets owing to a local modification of the field which accelerates the electron, there is some uncertainty as to the way in which we should represent mathematically the resulting electromagnetic field when the light is of the frequency of visible light or even X-rays, for a large number of doublets may be emitted in a single period. It seems best at first to simplify matters by considering first fields in which doublets or charges are emitted in all directions and then some fields in which doublets and charges are emitted only in a few particular directions.

Let us first of all consider the field ($\boldsymbol{E}$, $\boldsymbol{H}$) given by

$$\boldsymbol{H} + i\boldsymbol{E} = \nabla\sigma \times \nabla\tau + \frac{i}{c}\left[\frac{\partial\tau}{\partial t}\nabla\sigma - \frac{\partial\sigma}{\partial t}\nabla\tau\right],$$

where $r = c(t - \tau)$ is the distance of a point P whose coordinates are (x, y, z) from a moving point Q whose coordinates at time τ are (ξ, η, ζ) and whose velocity is $\boldsymbol{v}$. The quantity σ is given by the equation

$$\sigma = \frac{(\boldsymbol{l}\cdot\boldsymbol{r}) + k}{(\boldsymbol{v}\cdot\boldsymbol{r}) - cr},$$

where $\boldsymbol{l}$ is a complex vector which is a function of τ and k is a function of τ. The vector QP is denoted by $\boldsymbol{r}$.

In this type of field both electric and magnetic charges are emitted in all directions from the moving pole Q and the electric charge associated with the pole may vary. When $\boldsymbol{l} = e\boldsymbol{v}'$ and $k = e(c^2 - v^2)$, we obtain the field of a moving electric pole from which no charges are emitted. If $k = e(c^2 - v^2)$ and the vector function $\boldsymbol{l}$ changes sign periodically or represents a rotating vector the electric and magnetic charges which are emitted may be regarded as forming doublets.

The lines of electric force may be considered as made up of light-particles projected from the pole Q when k is real. The differential equation satisfied by the unit vector $\boldsymbol{s}$ which indicates the direction of projection at time τ is

$$k\boldsymbol{s}' = \boldsymbol{p} + (\boldsymbol{s} \times \boldsymbol{q}) - \boldsymbol{s}(\boldsymbol{s} \cdot \boldsymbol{p}),$$

where

$$\boldsymbol{p} + i\boldsymbol{q} = c\boldsymbol{l} + i(\boldsymbol{v} \times \boldsymbol{l})$$

and both $\boldsymbol{p}$ and $\boldsymbol{q}$ are real vectors.

We are interested in the case when a line of force returns periodically to its original form. Let us consider uniform circular motion and write $\boldsymbol{v} \equiv (v \cos \omega\tau, \quad v \sin \omega\tau, \quad 0)$, $\boldsymbol{l} \equiv (- e\omega v \sin \omega\tau, e\omega v \cos \omega\tau, ib)$; then

$$\boldsymbol{p} \equiv [- (b + ec\omega)v \sin \omega\tau, (b + ec\omega)v \cos \omega\tau, 0],$$

$$\boldsymbol{q} \equiv (0, 0, cb + e\omega v^2).$$

It is easy to see that if $\boldsymbol{p} \neq 0$ the differential equation for $\boldsymbol{s}$ possesses only a finite number of solutions of type

$$\boldsymbol{s} = \frac{\alpha + \beta \cos \omega\tau + \gamma \sin \omega\tau}{f + g \cos \omega\tau + h \sin \omega\tau}$$

and that there are consequently only a finite number of lines of force which return to their original form after a period of time $2\pi/\omega$. In the case when $\boldsymbol{p} = 0$ and $k = e(c^2 - v^2)$ the differential equation for $\boldsymbol{s}$ reduces simply to

$$\frac{d\boldsymbol{s}}{d\tau} = \omega(\boldsymbol{n} \times \boldsymbol{s}),$$

where $\boldsymbol{n}$ is a unit vector in the direction of the axis of z. All the solutions of this equation are periodic with a period $2\pi/\omega$ and so all the lines of force return to their original form periodically. The field in this case is of the type considered by Leigh Page.*

As an example of a field in which charges are fired out in particular directions let us consider the following expressions for the components of $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$E_x = \frac{ex}{r^3} + \frac{\lambda y}{x^2+y^2} - \frac{\nu z x}{r(x^2+y^2)},$$

$$E_y = \frac{ey}{r^3} - \frac{\lambda x}{x^2+y^2} - \frac{\nu z y}{r(x^2+y^2)}, \qquad E_z = \frac{ez}{r^3} + \frac{\nu}{r},$$

$$H_x = \frac{\lambda z x}{r(x^2+y^2)} + \frac{\nu y}{x^2+y^2}, \qquad H_y = \frac{\lambda z y}{r(x^2+y^2)} \cdot \frac{\nu x}{x^2+y^2},$$

$$H_z = -\frac{\lambda}{r},$$

where λ and ν are functions of $\tau = t - r/c$.

We may write $\boldsymbol{E} = rd\boldsymbol{s} - c\boldsymbol{s}d\tau$, where $\boldsymbol{s}$ is a unit vector with components (l, m, n) which at time τ is in the direction of the radius vector r, provided

$$\frac{dl}{d\tau} = \frac{c\nu}{e}\,\frac{ln}{l^2+m^2} - \frac{c\lambda}{e}\,\frac{m}{l^2+m^2},$$

$$\frac{dm}{d\tau} = \frac{c\nu}{e}\,\frac{mn}{l^2+m^2} + \frac{c\lambda}{e}\,\frac{l}{l^2+m^2},$$

$$\frac{dn}{d\tau} = -\frac{c\nu}{e}.$$

These equations give

$$\frac{d}{d\tau}(l+im) = \frac{c\nu}{e}\,\frac{n(l+im)}{1-n^2} + \frac{ic\lambda}{e}\,\frac{l+im}{1-n^2},$$

consequently when n has been expressed in terms of τ, $l+im$ may be found by a simple integration. It may be noticed that when $\nu = 0$ and λ is a constant the lines of force revolve around the axis of z at different rates. If θ denote the angle which a line of force makes with the axis of z and ϕ denote

* *Proc. Nat. Acad. of Sci.*, March, 1920, p. 115.

its angular velocity around this axis we have $\sin^2\theta\cdot\dot{\phi} = c/e$. If each line of force carried a unit mass at unit distance from the origin the angular momentum of the mass would thus be the same for all the lines of force.

An attempt made previously* to obtain the lines of force when any system of charges or doublets is emitted from a moving pole is vitiated by an unfortunate oversight. It appears that Z cannot be made equal to unity as was assumed. The differential equations to be solved are consequently of type

$$g(\sigma, \bar{\sigma})\frac{d\sigma}{d\tau} = f'(\bar{\sigma}, \tau), \qquad g(\sigma, \bar{\sigma})\frac{d\bar{\sigma}}{d\tau} = f'(\sigma, \tau),$$

where g is a function† whose form is independent of the emitted system and consequently independent of the form of f'.

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SHORTER NOTICES.

Table de Charactéristiques de Base 30030, *donnant en un seul coup d'oeil les facteurs premiers des nombres premiers avec* 30030 *et inférieurs à* 901,800,900. By Ernest Lebon. Tome I, Premier Fascicule. Paris, Gauthier-Villars, 1920.

To one who has spent eight years of his life in making a factor-table for the first ten millions the plan to extend such a table to the limit 901,800,900 seems like a rather serious undertaking. If such a table were constructed according to the plan devised by Burckhardt and employed by Dase and Glaisher the number of pages would exceed one hundred thousand, and with five hundred pages to the volume would fill some two hundred volumes! If, as in the tables published by the Carnegie Institute, the multiples of 2, 3, 5 and 7 were omitted, and the pages somewhat enlarged to take care of the large divisors certain to appear, the number of pages would be

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* *Proc. London Math. Soc.*, (2), vol. 18 (1919), p. 123, *Phil. Mag.*, vol. 34, Nov., 1917, p. 419. In the notation used in the last paper the equations should be

$$g(\alpha, \beta)\frac{d\alpha}{d\tau} = f(\beta, \tau), \qquad g(\alpha, \beta)\frac{d\beta}{d\tau} = f(\alpha, \tau).$$

† In the case of a stationary pole, $g = (1 + \sigma\bar{\sigma})^{-2}$.