By mathematical induction it is proved that the leading term in

$$\sum_{k=0}^{n} \binom{2n}{n+k} k^{2p+1}$$

is

$$\frac{1}{2} p!\, n^{p+1} \binom{2n}{n}.$$

Corresponding to the application made at the end of § 1, we have here

$$\begin{aligned} \int_0^\infty e^{-x^2/2\sigma^2} x^{2p+1} dx &= \lim_{n=\infty} \frac{1}{\binom{2n}{n}} \sum_{k=0}^{\infty} \binom{2n}{n+k} (k\Delta x)^{2p+1} \Delta x \\ &= \lim_{n=\infty} \frac{1}{2} p!\, n^{p+1} \left(\frac{2\sigma^2}{n}\right)^{p+1} \\ &= \frac{1}{2} p!\, (2\sigma^2)^{p+1}. \end{aligned}$$

For example, since the area under the whole curve $y = e^{-x^2/2\sigma^2}$ is $\sigma\sqrt{2\pi}$, the "mean deviation" of this area is $\sigma\sqrt{2/\pi}$.

The products of the binomial coefficients by powers of terms of other arithmetical progressions do not seem to give simple results analogous to those obtained by Kenyon; this question is reserved for further study.

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THE WORK OF POINCARÉ ON AUTOMORPHIC FUNCTIONS.

Oeuvres de Henri Poincaré, publiées sous les auspices du Ministère de l'Instruction publique par G. DARBOUX. Tome II, publié avec la collaboration de N. E. NÖRLUND et de ERNEST LEBON. Paris, Gauthier-Villars, 1916. lxxi + 632 pp.

THE collected works of Poincaré will fill some 10 volumes, of which the one before us is the first to be published. It contains the principal papers written by him in the field of automorphic functions, where some of his most brilliant early work lies. According to the statement of Darboux in the preface, this Volume II appears first with the hope that it may stimulate mathematicians to active work in that field.

An invaluable and necessary revision with critical notes is supplied by Nörlund.

The many remarkable eulogies pronounced shortly after the death of Poincaré testify to a very wide recognition of his dominating position in the mathematical world. In one of the best of these, Volterra says: "if we were to characterize the recent period of the history of mathematics by a single name we should all give that of Poincaré."* Among these appreciations that of Hadamard† may be mentioned here for its critical value, while the admirable "Eloge Historique" of Darboux deserves its introductory place in the volume.

It is impossible to give any satisfactory idea of the achievements of Poincaré in the space of a single essay. The appearance of his collected papers arranged by subjects will furnish an occasion for a more leisurely and critical review of his work in its relation to the mathematics of the time. It is my purpose to attempt such a review.

An immediate stimulus for Poincaré's researches in the domain of automorphic functions was the question proposed for the Grand Prix of 1880 in the mathematical sciences: to complete in some important point the theory of ordinary linear differential equations. Partly on account of incompleteness in his development of the new functions, the prize was not awarded to Poincaré but to G. H. Halphen. An extract from this initial attempt of Poincaré is to appear in volume 39 of the *Acta Mathematica*.

In order to understand the nature of the advance made by Poincaré it is necessary to go back to the nearly contemporary work of Schwarz, Fuchs, Schottky, and Klein.

It was obvious after Riemann's time that the inverse of the ratio of a pair of solutions of an ordinary linear differential equation of the second order was an automorphic function, i.e., one unaltered by a group of linear fractional transformations. Examples of such functions were at hand

* Volterra, "Henri Poincaré." A lecture delivered at the inauguration of Rice Institute. Translated by G. C. Evans. Rice Institute Pamphlets, Vol. I (1915), No. 2, pp. 133–162.

† "Henri Poincaré: le mathématicien," *Revue de Métaphysique et de Morale*, vol. 21 (1913), pp. 617–658.

in the trigonometric and elliptic functions, the elliptic modular functions, the more general triangle functions first classified by Schwarz, other functions due to Schottky, and finally the rational automorphic functions whose beautiful algebraic and geometric relations were developed by Klein. When considered in relation to the illuminating work of Fuchs on ordinary linear differential equations, these examples made it an obvious probability that an extensive theory of single-valued transcendental automorphic functions awaited development. Fuchs pointed out this field explicitly first in 1880, but Poincaré's account of the genesis of his ideas indicates his essential independence of this paper.*

The general transcendental automorphic functions, however, were not immediately approached, because of a certain gap to which we will now refer.

If we take the singular points of the differential equation and the characteristic exponents as real, it is an obvious deduction from the work of Fuchs before 1880 that the inverse function maps a circular polygon conformally upon a Riemann's surface, and that by the process of analytic extension other polygons which are obtained by linear fractional transformations, arise. An immediate necessary condition that there is no overlapping of the polygons is that the angles at the vertices of the first polygon are either 0 or fractional parts of 2π. The precise further conditions for non-overlapping were known only in the case of the rational automorphic functions, treated by Klein, where the polygons were representable as spherical polygons so that the ordinary formulas of spherical trigonometry were available. But the earlier work of Fuchs made it clear that single-valued automorphic functions exist always when there is no overlapping.

Thus the moment at which Poincaré reached scientific maturity was a most favorable one in which to create a general theory of these functions which were challenging the attention of mathematicians.

Here it seems worth while to call attention to a situation which often exists in mathematics but scarcely ever exists in any other field of science. It is sometimes the case that several mathematicians are in possession of much new material save for a single link of reasoning. However, on

* "Science et Méthode," Flammarion, Paris, 1908, Chap. 3; or pp. lvii–lviii of Darboux's "Éloge Historique."

account of self-imposed requirements of completeness and rigor, no claim is made for credit.

It was precisely such a state of affairs that existed in the domain of the transcendental automorphic functions when Poincaré began his work. The algebraic relation between automorphic functions with the same group, the uniformizing properties of these functions, and their use in solving linear differential equations are illustrations of material of this sort.

It was by his use of non-euclidean geometry that he overcame the one fundamental difficulty referred to above. How then did these geometric ideas enter?

The region of the complex plane under consideration is the interior of a fixed circle and the bounding circles of the polygons referred to are all orthogonal to it. Furthermore, the linear fractional transformations of these polygons are such as to leave the fixed circle invariant. Here, then, are the abstract features of the non-euclidean geometry of Lobachevski, namely, a three-parameter group of transformations which may be interpreted as the group of rigid motions if every circle orthogonal to the fixed circle be regarded as a straight line, and if angles be interpreted as usual.

In recognizing this fact Poincaré faced an extensive vista of possible developments in the field of automorphic functions, and incidentally obtained a simple new geometric representation of the hyperbolic plane. It was natural to connect this geometric view with the highly suggestive work of Klein. Thus we are not surprised to find Poincaré a master of the powerful weapons furnished by projective geometry at the outset.

From this new point of view the difficulty of overlapping presents an absolutely concrete aspect. The polygon appears as an ordinary polygon in the non-euclidean plane, and the problem is to determine when the entire plane can be filled by a network of congruent polygons. The precise analytical conditions are seen to be algebraic.

Although the essential kernel of the advance of Poincaré was only a single step, he must be regarded as the true founder of the theory of the general transcendental automorphic function. A somewhat analogous situation is presented by the founding of the calculus. Without doubt many mathematicians were in possession of most of the essential ideas of the calculus, but failed to discern the possibility of elevating it to an independent discipline by the invention of a suitable symbolism. Nevertheless there is general agreement that Newton and Leibniz were the originators of the calculus, precisely because they did see this possibility.

Had Newton merely developed the simpler parts of the calculus of fluxions, his title to preeminence would have lost some of its validity. But Newton not only conceived the possibility but carried it to fulfilment by developing an adequate technique and applying it to the problems of celestial mechanics. Likewise, Poincaré merits great credit because, once in possession of the fundamental new weapon of attack, he proceeded to develop the theory of the transcendental automorphic functions to a remarkable degree.

In the further development of his new ideas Poincaré faced two possibilities. On the one hand, following the general course of Riemann, Schwarz, Fuchs, Schottky, and Klein, he might develop a general theory based on existence theorems for conformal mapping. This was a possible course as he saw plainly. Thus he declares in an early paper (*Mathematische Annalen*, 1882, translation) "From the existence of discontinuous groups one might without doubt, by processes analogous to those of Schwarz, deduce that of single-valued functions reproduced by the substitutions of the group; but one would have then no explicit expression for these transcendental functions. Moreover it is better to use other methods." The alternative method was based upon certain explicit series invented by Poincaré before he noted the geometric relation outlined above, and to these series we now turn.

In the case of an ordinary finite group of linear fractional transformations the sum of a rational function $H(z)$ of z, taken for any value of z and its transformed arguments z_k under the group, clearly forms an invariant function. Similar infinite series exist in the case of the series for the elliptic functions. These infinite series, invented by Cayley and Eisenstein, diverge in certain cases although their derivative series converge. On the basis of such series Weierstrass constructed his theory of the elliptic functions.

Suppose that in an analogous way we form the series formally invariant under an arbitrary infinite discontinuous group of linear fractional transformations leaving invariant a circle. The image points z_k of z will approach the invariant circle, as k becomes infinite, and thus the kth term $H(z_k)$ of the series will remain finite, at least if the rational function H with which we start has no poles upon that circle. But there is no reason to assume that this series converges.

If, however, the series be formally differentiated term by term, the typical terms of the new series are the product of the derivative rational function $H'(z_k)$ taken at the image point z_k and the derivative dz_k/dz. The modulus of the second factor represents the ratio of arc lengths of corresponding similar infinitesimal figures. The area integral of the squared modulus of this factor over any part C of a fundamental region yields the ordinary area of the image C_k. But the sum of all the areas C_k is less than the area of the invariant circle. Hence the series

$$\Sigma \int\int \text{mod } (dz_k/dz)^2 dxdy$$

converges. An easy extension of this argument shows that $\Sigma \text{mod}(dz_k/dz)^2$ also converges. Thus the factor dz_k/dz approaches 0 as k becomes infinite.

Now if such a derivative series converges it does not represent an automorphic function but a function I with the property

$$I(z_k) = I(z)(c_k z + d_k)^{-2},$$

where

$$z_k = (a_k z + b_k)/(c_k z + d_k)$$

with $a_k d_k - b_k c_k = 1$. Thus one is led to consider series whose typical term is of the form $H(z_k)\ (dz_k/dz)^m$ which will converge if $m \geqq 2$, and which are multiplied by $(c_k z + d_k)^{-2m}$ when z is changed to z_k.

Moreover the functions defined by these theta series of Poincaré play the rôle of the elliptic theta functions for elliptic functions and are suited to form a basis for the general development of the theory of automorphic functions; in fact the quotient of two such functions having the same value of m is an automorphic function. No other completely explicit expressions for the automorphic functions have as yet been found.

All of these facts can be looked at from the point of view of homogeneous variables and then take on a more elegant form, but it seems probable that the development of Poincaré's own ideas concerning these series was approximately along the line of least resistance just outlined.

With these two touchstones of the theory in his possession—namely, the interpretations of non-euclidean and projective geometry, and the explicit series above mentioned—the analogy of the theory of automorphic functions and the elliptic functions became doubly apparent. The interrelation of elliptic functions with algebraic functions and their integrals, with linear differential equations, and with number theory were seen by him to admit of complete generalization, and the range of knowledge called for in order to carry out this development was just that which Poincaré had come naturally into possession of at Paris.

His five long papers in the early volumes of the *Acta Mathematica* give a first approximation to the theory of the automorphic functions even at its present degree of development. A large variety of important complementary results have been added, mainly by German mathematicians, who have used homogeneous variables, and have developed an independent theory from the point of view of Riemann alluded to above.

In the first of these extensive papers entitled "Théorie des groupes fuchsiens," Poincaré developed explicitly the forms of polygons and networks of congruent polygons in the non-euclidean plane from his geometric point of view. He called discontinuous groups which possess an invariant circle Fuchsian groups in honor of Fuchs, and the corresponding automorphic functions Fuchsian functions. Perhaps it is better to use the terminology of Klein, and speak of groups with principal circle and automorphic functions with principal circle. Poincaré's classification of the types of groups is suggestive but has not the definitive form which has been given it by Fricke.

In a second paper "Sur les fonctions fuchsiennes" he considers the series above obtained and develops the main facts concerning the functions which they define. The chief difficulty is caused by the fact that such a function may vanish identically. By means of the theory of these functions a theory of the automorphic functions can also be derived. A cardinal fact to prove here is that every automorphic function can be represented as the quotient of two such series. Conversely, one can pass from the automorphic function back to the theta series of Poincaré by a process of differentiation.

A first fundamental application of these functions, noted in the same article, is to the uniformization of algebraic functions. Between any pair of automorphic functions with the same group an algebraic relation obviously exists, just as an algebraic relation exists between two doubly periodic functions with the same period parallelogram. The automorphic functions uniformize this algebraic relation in the sense that each variable is expressible by means of an automorphic function belonging to the same group. In this way the Riemann surface is mapped conformally upon the circular polygon referred to earlier. Thus the important question arises at once: is it not possible to uniformize every algebraic relation by means of such function? An early count of constants showed Poincaré that this was in all probability the case. A second application of importance lay in the integration of the corresponding linear differential equation of the second order.

In a third paper "Mémoire sur les groupes kleinéens" Poincaré treated the more general automorphic group in which there was no invariant circle, when the same theta series were available. These groups and the corresponding automorphic functions were called Kleinian by him, but are perhaps better designated as groups and automorphic functions without principal circle. The fundamental geometric results which made an attack on these more general functions possible were due to Cayley and Klein. Spatial non-euclidean geometry enters, inasmuch as the totality of linear fractional transformations of the complex plane is isomorphic with those projective transformations of a quadric in space which leave the quadric invariant.

In two further papers, "Sur les groupes des équations linéaires" and "Mémoire sur les fonctions zétafuchsiennes," Poincaré attacks the problem of uniformizing algebraic functions and the integrals of linear differential equations with algebraic coefficients. It is easy to see that if the function has only three branch points or if the linear differential equation has rational coefficients and three singular points, regular or irregular, then such a uniformization is possible by means of the elliptic modular function, provided the images of the three singular points lie at the vertices of the fundamental triangle. This admits of obvious generalizations. In his attempt to establish such generalizations he proves first that the characteristic constants of the monodromic group are entire functions of the literal coefficients, and then employs the "method of continuity" of Klein. The satisfactory application of this method has proved difficult. Poincaré gave explicit series analogous to his theta series in terms of which such uniformization for the differential equation is possible.

In attempting to appraise the importance of the new functions discovered by Poincaré, one must always remember that these functions form a natural extension of many of the functions treated earlier in analysis. Moreover, they afford a very simple systematic method of approach to the theory of the algebraic functions and their integrals, and add an illuminating chapter in the theory of ordinary linear differential equations with algebraic coefficients. But the process of natural generalization appears to terminate with the invention and investigation of these functions, just as the theory of the elliptic functions reached its conclusion earlier. I do not expect to see large further developments take place, notwithstanding the hope to the contrary implied by Darboux in the preface.

The mathematician of the future, having before him such completed theories, will seek on the one hand to clarify and simplify, and on the other hand to use only as much of them as is really vitally related to further advance.*

If this be true it may be predicted that the elliptic modular functions and the triangle functions will maintain an important position on account of their intrinsic importance and as the best examples of transcendental automorphic functions with principal circle, while the more general functions discovered by Poincaré will receive only such consideration as is necessary to establish their position in the hierarchy of analytic functions.

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* A treatment of automorphic functions in this spirit is given in the concluding chapter of Professor Osgood's "Lehrbuch der Funktionentheorie," vol. 1 (second edition), Teubner, Leipzig, 1912. See also his paper "On the uniformization of algebraic functions," *Annals of Mathematics*, ser. 2, vol. 14 (1912–1913), pp. 143–162.