THE TWENTY-SIXTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-sixth summer meeting of the Society was held at the University of Michigan, Ann Arbor, Mich., September 2–4, 1919, in conjunction with meetings of the Mathematical Association of America and the American Astronomical Society. At a joint dinner of the three organizations on Thursday evening Mr. J. E. Beal, of the board of regents of the University, and Professor W. W. Beman, in the absence of the president and the dean, made addresses of welcome to the visiting societies, to which Professors Schlesinger, Eisenhart and Rietz gave appropriate responses in behalf of the societies; and a formal vote of thanks for the generous hospitality of the university was adopted and subsequently presented to the proper authorities.

The Michigan Union and Newberry Residence were opened for the accommodation of the visitors, meals being served in the former building. A complimentary luncheon given by the University on Thursday, an informal reception by Professor and Mrs. Hussey at the Detroit Observatory on Tuesday evening, and auto rides around Ann Arbor and to the neighboring city of Detroit were only three of the many kindnesses of the hosts. An exhibition of the astronomical and mathematical rarities in the University of Michigan Library attracted much attention.

A meeting of the Council of the Society was held Wednesday morning in Alumni Memorial Hall. Dr. Raymond W. Brink, of the University of Minnesota, was elected to membership in the Society, and six new applications for membership were received.

At the joint dinner on Thursday evening there were one hundred and eighty persons in attendance. At the regular scientific sessions of the Society there was an attendance of over a hundred, including the following eighty members of the Society:

Dr. E. S. Allen, Professor R. C. Archibald, Professor G. N. Armstrong, Professor W. W. Beman, Professor Susan R. Benedict, Professor G. A. Bliss, Professor Henry Blumberg, Professor R. L. Borger, Professor J. W. Bradshaw, Professor

W. C. Brenke, Professor E. W. Brown, Professor W. H. Butts, Professor W. D. Cairns, Professor A. L. Candy, Professor E. H. Clarke, Dr. G. H. Cresse, Professor D. R. Curtiss, Professor S. C. Davisson, Professor F. F. Decker, Professor L. L. Dines, Professor L. W. Dowling, Professor L. P. Eisenhart, Professor John Eiesland, Professor L. C. Emmons, Professor G. C. Evans, Professor Peter Field, Professor B. F. Finkel, Professor W. B. Ford, Mr. T. C. Fry, Dr. Elizabeth B. Grennan, Professor C. F. Gummer, Professor A. G. Hall, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. A. Hopkins, Professor H. A. Howe, Professor E. V. Huntington, Professor W. J. Hussey, Professor L. C. Karpinski, Professor A. J. Kempner, Professor A. M. Kenyon, Professor H. R. Kingston, Professor Florence P. Lewis, Professor G. H. Ling, Professor A. C. Lunn, Dr. E. B. Lytle, Professor J. V. McKelvey, Professor W. D. MacMillan, Professor J. L. Markley, Professor G. A. Miller, Professor E. J. Moulton, Professor F. R. Moulton, Dr. A. L. Nelson, Professor F. W. Owens, Professor C. I. Palmer, Professor A. D. Pitcher, Professor L. C. Plant, Professor V. C. Poor, Professor R. G. D. Richardson, Professor H. L. Rietz, Dr. R. B. Robbins, Professor Maria M. Roberts, Professor E. D. Roe, Jr., Mrs. E. D. Roe, Professor W. H. Roever, Professor G. T. Sellew, Professor J. B. Shaw, Professor H. E. Slaught, Professor G. W. Smith, Professor P. F. Smith, Professor Virgil Snyder, Professor R. P. Stephens, Professor E. B. Stouffer, Professor A. L. Underhill, Professor J. N. Van der Vries, Mr. H. E. Webb, Professor K. P. Williams, Professor B. F. Yanney, Professor J. W. Young, Professor Alexander Ziwet.

Thursday afternoon was devoted to a joint scientific session, at which Professor Slaught, President of the Mathematical Association, presided. The following addresses were given:

1. "Mathematics and statistics," by Professor E. V. Huntington, being the retiring address of the former president of the Mathematical Association.

2. "The work of the National Research Council with reference to mathematics and astronomy," by Professor E. W. Brown.

3. "Report on the international conference of scientists at Brussels," by Professor Frank Schlesinger, President-elect of the American Astronomical Society.

Professor Huntington gave an introductory account of the mathematics of statistics and urged the mathematical fraternity to give the subject more consideration than it has received in the past. His address will appear in full in the *American Mathematical Monthly* in November.

Professor Brown commented on the lack of close cooperation, previous to the period of the war, amongst the various scientific societies of our country. About a year before our entry into the war the National Research Council was organized under the enthusiastic and capable leadership of Dr. G. E. Hale, partly as a preparedness measure, with the realization that in case of war cooperation would be imperative for efficient service from our scientific men. After our entry into the war innumerable problems of a scientific character were presented to the Council and through its aid were effectively treated. Now, in time of peace, it aims to encourage research and to aid in the promotion of science and scientific interests where a broad cooperation is desirable for best results. For much of its work the Council is divided, in the tentative plans, into a number of divisions, mathematics being associated with the physical sciences group, along with astronomy, physics and geophysics. Professor Brown emphasized the fact that the National Research Council did not wish in any way to interfere with the freedom of action of any of our scientific societies, or of individual scientists, but hoped primarily to aid in matters when cooperation was obviously desirable. Concrete illustrations of possible lines of activity were suggested, including the provision, with international cooperation, of bibliographies of science and the selection of a standard system of units and notation, the propaganda for the education of people to an appreciation of the importance of science, and the securing of government aid in the development of scientific projects.

Professor Schlesinger reported the organization of and results accomplished by the International Research Council which met at Brussels, July 18–30. American astronomers took an active part in the conference but no American mathematicians were present. Professor Schlesinger gave an account of the activities of the party of American astronomers and the organization of an International Astronomical Union. An extended report of the paper will be published in *Popular Astronomy*. Dr. L. A. Bauer, of the Carnegie Institution, who also attended the Brussels conference, spoke briefly of the organization of an International Geophysical Union as well as the International Astronomical Union.

At the regular sessions of the Society for the presentation of research papers, held Tuesday afternoon, Wednesday morning and afternoon and Thursday morning, Professors Beman, Curtiss, Snyder and Bliss served as presiding officers. Professor Karpinski served as secretary at the first session, and Professor E. J. Moulton at the other sessions. The following thirty-three papers were read:

(1) Professor PETER FIELD: "On wind corrections."

(2) Professor H. J. ETTLINGER: "Cauchy's memoir of 1814 on definite integrals."

(3) Mr. L. H. RICE: "Expansion of any determinant in minors from rectangular panels."

(4) Dr. A. L. NELSON: "Pseudo-canonical forms and invariants of systems of partial differential equations."

(5) Professor A. J. KEMPNER: "On the separation of complex roots of an algebraic equation."

(6) Dr. C. N. REYNOLDS, JR.: "Some theorems on the zeros of solutions of homogeneous linear differential equations of the nth order."

(7) Dr. REYNOLDS: "Some theorems on the zeros of solutions of self-adjoint homogeneous linear differential equations of the fifth order."

(8) Dr. J. W. ALEXANDER: "Proof of the existence of distinct three-dimensional manifolds with the same group."

(9) Professor E. D. ROE, JR.: "Certain determinants expressible as circulants or skew-circulants."

(10) Professor W. B. FORD: "A brief account of the life and work of the late Professor Ulisse Dini."

(11) Dr. S. P. SHUGERT: "The resolvents of König and other types of symmetric functions."

(12) Professor G. A. MILLER: "Form of the number of subgroups of prime-power groups."

(13) Dr. E. S. ALLEN: "A generalization of a formula of Schubert in enumerative geometry."

(14) Dr. E. P. LANE: "Joint-axis congruences with indeterminate developables."

(15) Dr. R. W. BRINK: "A modification of an integral test for the convergence and divergence of infinite series."

(16) Professors F. R. SHARPE and VIRGIL SNYDER: "Certain types of involutorial space transformations (second paper)."

(17) Professor L. P. EISENHART: "Transformations of surfaces applicable to a quadric."

(18) Professor EISENHART: "Transformations of cyclic systems of circles."

(19) Professor G. A. BLISS: "Differential equations containing arbitrary functions."

(20) Professor BLISS: "Functions of lines in ballistics."

(21) Professor D. R. CURTISS: "On the relative positions of the complex roots of an algebraic equation with real coefficients and those of its derived equation."

(22) Professor H. L. RIETZ: "Urn schemata as a basis for the development of the theory of correlation."

(23) Professor L. L. DINES: "Projective transformations in function space (second paper)."

(24) Professor J. B. SHAW: "On the invariants belonging to a hypernumber in an algebra of infinite order."

(25) Professor R. D. CARMICHAEL: "Conditions necessary and sufficient for the existence of a Stieltjes integral."

(26) Professor CARMICHAEL: "Note on convergence tests for series and on Stieltjes integration by parts."

(27) Professor CARMICHAEL: "Note on a physical interpretation of Stieltjes integrals."

(28) Professor H. A. HOWE: "An apparent anomaly in errors of interpolated values."

(29) Professor G. C. EVANS: "Transformations of a Stieltjes integral potential."

(30) Professor T. H. HILDEBRANDT: "Note on sequences of Stieltjes integrals."

(31) Professor W. H. ROEVER: "Equations of motion of a projectile regarded as a particle."

(32) Professor W. D. CAIRNS: "Certain properties of binomial coefficients."

(33) Professor A. J. KEMPNER: "On the shape of polynomial curves."

In the absence of the authors the papers by Professor Ettlinger, Mr. Rice, Dr. Reynolds, Dr. Alexander, Dr. Shugert, Dr. Lane, Dr. Brink, and Professor Carmichael, were read by title only. The papers on ballistics led to a somewhat extended discussion by Professors Field, Bliss, Roever, and F. R. Moulton. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles above:

1. Professor Field's paper deals with the fundamental assumptions on which the ordinary wind correction formulas are based. It will be published in the *Journal of the U. S. Artillery.*

2. In this paper Professor Ettlinger discusses Cauchy's memoir of 1814 from a historical and critical point of view. Eminent mathematicians of that day, such as Poisson, Lacroix and Legendre, did not properly appreciate the importance of the contribution and merely saw in the memoir new methods of evaluating definite integrals which Euler had computed years before. For the modern reader, also, it is difficult to find in this memoir the very first statement of Cauchy's discoveries on residues and the integral theorem upon which the structure of the modern theory of functions of a complex variable is built. These facts the author brings into the light by simplifications, both algebraic and geometrical. The method is also made to conform to our modern standard of rigor.

The subject of this paper was suggested by Professor W. F. Osgood.

3. By availing himself of the full force of a new proof of Muir's recent theorem (*Messenger of Mathematics*, May, 1918) that "a determinant can be expressed in terms of minors drawn from four mutually exclusive arrays, two of which are coaxial and complementary to one another," Mr. Rice shows, not only that the last clause of the theorem can be freed of the restriction to coaxial (square) arrays, but also that there need not be precisely four rectangular arrays or panels but the matrix of the determinant may be subdivided into any number of panels independent of one another in point of dimensions, complementary sets of minors being drawn from these panels.

4. In a recent note (*American Journal of Mathematics,* volume 41, April, 1919, pages 123–132), Dr. Nelson has shown that fundamental sets of seminvariants of a completely integrable system of linear homogeneous partial differential equations may be found by the reduction of the system to certain pseudo-canonical forms. The method is here extended to include semi-covariants, invariants and covariants. Simul-

taneous invariants of several such completely integrable systems may be similarly constructed. Application is also made to semi-covariants of algebraic forms.

5. By simple vector considerations Professor Kempner determines certain sectors starting at the origin in the complex plane which do not contain roots of a given algebraic equation. These sectors depend, in case the equation has real coefficients, only on the sequence of signs of the coefficients; in case the equation has complex coefficients, only on the arguments of the coefficients. When the equation is of low degree or when, while the degree of the equation is high, many coefficients vanish, the information obtained is often sufficient to separate entirely the complex roots. In most cases the information is less precise. The method applies also to equations containing fractional powers of the unknown.

6. In this paper Dr. Reynolds proves a general separation theorem for the zeros of solutions of homogeneous linear differential equations of the nth order and compares the zeros of the solutions of the two differential systems

$$(1)\quad \begin{aligned} &y^{(n)}(x) + \sum_{i=2}^{n} p_i(x) y^{(n-i)}(x) = 0, \\ &\qquad y^{(k)}(\xi) = \alpha_k \quad (k = 0, 1, 2, 3, \cdots, n-1), \end{aligned}$$

$$(2)\quad \begin{aligned} &\eta^{(n)}(x) + \sum_{i=2}^{n-1} p_i(x) \eta^{(n-i)}(x) + [p_n(x) + R(x)]\eta(x) = 0, \\ &\qquad \eta^{(k)}(\xi) = \alpha_k \quad (k = 0, 1, 2, 3, \cdots, n-1). \end{aligned}$$

For n even the theorems obtained generalize the author's earlier results concerning equations of the fourth order. For n odd the theorems obtained generalize Professor Birkhoff's results concerning equations of the third order (*Annals of Mathematics,* second series, volume 12).

7. Dr. Reynolds applies the general separation theorem of the preceding paper to the solutions of self-adjoint homogeneous linear differential equations of the fifth order. Any solution of such an equation can be expressed linearly in terms of the six functions $\eta_i\eta_j' - \eta_j\eta_i'$ $(i, j = 1, 2, 3, 4; i < j)$, where the η's satisfy a homogeneous self-adjoint linear differential

equation of the fourth order. The wronskian of any two solutions of the fifth order equation is a homogeneous quadratic form in the η's. By means of these two relations the author proves a group of theorems concerning the solutions of self-adjoint equations of the fifth order which are of the same degree of generality as his earlier results concerning self-adjoint equations of the fourth order and interprets his results geometrically by means of the integral curve of the self-adjoint equation of the fourth order which is satisfied by the η's.

8. In this paper, Dr. Alexander shows that two distinct three-dimensional manifolds with the same Betti numbers, coefficients of torsion, and group exist. The Heegaard diagrams of the manifolds consist of anchor rings on which have been traced the curves ab^5 and a^2b^5 respectively.

9. Generalizing, Noether and Puchta discovered determinants of m elements, and of order m, which were factorable into m linear factors of their elements, the factors being real for $m = 2^n$. Noether showed that, for $m = n_1n_2\cdots n_k$, the determinant is the simultaneous resultant of a first set of equations $x_1^{n_1} - 1 = 0$, $x_2^{n_2} - 1 = 0$, $\cdots$, $x_k^{n_k} - 1 = 0$, and $\Sigma a_{i_1i_2\cdots i_k}x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k} = 0$, where the degree of Σ in x_1, x_2, $\cdots$, x_k is $n_1 - 1$, $n_2 - 1$, $\cdots$, $n_k - 1$ respectively. Professor Roe extends the generalization to the resultant of a second set of equations $x_1^{n_1} + 1 = 0$, $x_2^{n_2} + 1 = 0$, $\cdots$, $x_k^{n_k} + 1 = 0$, and $\Sigma = 0$. He points out that by eliminating one variable x_j by Sylvester's method a circulant or a skew-circulant respectively, in the remaining variables, of order n_j results. The elimination of a second variable, not by Sylvester's method, but by forming the product of the function obtained by substituting all the values of that variable, obtained from its equation in the first or second set respectively, in the circulant or skew-circulant already formed, yields, by the laws for the multiplication of circulants or skew-circulants, a circulant or skew-circulant of order n_j, and containing one less variable. Continuation of the elimination in this way, step by step, gives the resultant finally as a circulant or skew-circulant of order n_j. As the variable x_j first eliminated was chosen at pleasure, it follows that the determinants of Noether and Puchta and their analogues

here suggested by Professor Roe can be expressed as circulants or skew-circulants respectively of any of the orders n_1, n_2, $\cdots$, n_k. Numerous identities follow from these different expressions. As a simple example of such identities, let $k = 2$, $n_1 = 2$, $n_2 = 3$. Then from one or the other form we can derive $A^3 + B^3 \equiv a^2 - b^2$, $A'^3 - B'^3 \equiv a'^2 + b'^2$, where A, B, A', B', a, b, a', b' are functions of 5 independent of 6 elements, another function C or C' being placed equal to zero to secure the form. The identities are satisfied in ∞^5 ways. They can be generalized to such as are satisfied in $\infty^{2n-1} = \infty^{3p-1}$ ways by $2n - 1$ independent elements, where $n \equiv 0 \pmod 3$, $p \equiv 0 \pmod 2$, and these in turn can be still further generalized. Professor Roe also considers the resultant of the elimination between a third set of equations $x_1^{n_1} - 1 = 0, \cdots, x^{n_k} - 1 = 0$, $x^{m_1} + 1 = 0, \cdots, x^{m_e} + 1 = 0$ and $\Sigma = 0$, by virtue of which the resultant can be shown to be a circulant of any of the orders n_1, $\cdots$, n_k or a skew-circulant of any of the orders m_1, $\cdots$, m_e. He also considers the resultant between any of the systems and $\Sigma = 0$, when all restrictions as to the degree of Σ are removed, and shows that according as $d_j \gtreqless n_j - 1$ (d_j the degree of x_j in Σ), circulants or skew-circulants with polynomial or zero elements result. Also such identities as $A^2 + B^2 \equiv A'^2 - B'^2$, $A^3 + B^3 \equiv A'^3 - B'^3$, etc.

10. Professor Ford's paper is intended to supplement the brief announcement of Professor Dini's death as recorded in the BULLETIN of last March and April.

11. Dr. Shugert's paper treats the following subjects, in part by methods and formulas derived or suggested by Professor Glenn in a paper on the theory of finite expansions in volume 15 of the *Transactions*: a lemma on determinants whose elements above the diagonal next above the principal diagonal are all zero; the expansion of a polynomial in ascending powers of a quadratic polynomial; conditions that a form contain a given quadratic form as a factor; separation of $a_x^m/(\xi_x^2)^k$ into partial fractions; the explicit formation, by determinants, of the resolvents studied by König in the *Mathematische Annalen*, volume 15 (1879); tables of symmetric functions of the sums and of the products, in pairs, of the roots of a given binary form. These tables are complete for the forms of orders 2 to 6 inclusive.

The paper is a University of Pennsylvania doctor dissertation and is being printed by the New Era Press by private arrangement with the author.

12. Professor Miller's paper appears in full in the present number of the BULLETIN.

13. If, on an algebraic curve of genus p, an algebraic series of point groups γ_m^1 of dimension 1 and a linear series g_n^r of any dimension r are given, the formula of Schubert states the number of those groups of γ_m^1 which contain $r + 1$ points of a single group of g_n^r.

Dr. Allen obtains the number of those groups of a series γ_m^2, of dimension 2, which contain $r + 2$ points of a single group of a given g_n^r. He obtains a recurrence formula by means of two pairs of correspondences between points of the curve, and from this finds the closed formula by mathematical induction.

The formula has the following two simple forms, which involve, respectively, the characters z_b of the series γ_m^2, and the numbers $d_{1[a]}$, of groups of γ_m^2 possessing a double points, together with enough arbitrary fixed points to make $d_{1[a]}$ result finite:

$$Z_{n,\, r;\, m,\, 2} = \sum_{b=0}^{2}\sum_{h=b}^{2}(-1)^{h+b}\binom{m-2-h}{r-h}\binom{n-r-h}{2-h}\binom{p-b}{h-b} z_b$$

$$= \sum_{a=0}^{2}\sum_{h=a}^{2}(-2)^{-a}\binom{m-2-a}{r-a}\binom{r-a}{h-a}\binom{n-r-h}{2-h} d_{1[a]}.$$

14. The joint-axis of a point P_y on a surface S_y, referred to a conjugate net, and of the corresponding point P_z on the first Laplace transform S_z of the net, has been defined by Wilczynski to be the line of intersection of the osculating plane of the curve $u =$ const. at P_y and the osculating plane of the curve $v =$ const. at P_z. Thus, all the joint-axes of pairs of corresponding points constitute a congruence. Dr. Lane studies the properties of the joint-axis congruence, and of the suite of Laplace transforms of S_y, which are characteristic of this configuration when the developables of the joint-axis congruence are indeterminate. He finds, by the methods of projective differential geometry, using the nota-

tion of Wilczynski's system (D), that there are two types of such configurations. In the first type, the joint-axes all pass through a fixed point, to which the third and minus second Laplace transforms of S_y reduce. In the second type, the joint-axes all lie in a fixed plane and give rise to a planar net of period three. The plane is characterized geometrically, and a number of geometrical theorems are obtained which relate certain points and lines in the configuration which have escaped notice hitherto.

15. In a previous paper Dr. Brink presented an integral test for the convergence and divergence of infinite series. It is applicable to series for which a simple analytic expression for a function $r(x)$ is known, where $r(n)$ is the ratio of the nth term of the series to the preceding term. And in this test $r(x)$ plays a rôle analogous to that played in the familiar Maclaurin-Cauchy integral test by the function $u(x)$, where $u(n)$ is the nth term of the series. In the present paper is given a modification of this test, in which a new form of the integral is introduced. This permits of a certain simplification of the conditions of the theorem, and a great simplification in the demonstration.

16. The paper of Professors Sharpe and Snyder is supplementary to that having the same title in the July *Transactions*. A general method of determining (1, 2) correspondences is developed and is applied to discuss the possible cases in which the order of the surfaces that are mapped on the planes of the double space does not exceed five. The discussion includes among others those involutions in which the surface of invariant points is birationally equivalent to a quartic with ten or more nodes.

17. In volume 19 of the *Transactions* Professor Eisenhart established a transformation of a pair of applicable surfaces into new pairs of applicable surfaces. In the present paper he shows that the transformations of surfaces applicable to a central quadric determined at great length by Guichard in the memoir which received the grand prize of the French academy in 1909 are of this kind, and that they follow readily from the general theory. Similar transformations of surfaces applicable to a paraboloid are established. The known trans-

formations of surfaces of constant curvature are likewise of this type. The general transformations of surfaces applicable to the quadrics admit two theorems of permutability, in consequence of which from known transformations others follow directly.

18. When the circles of a two-parameter family of circles are orthogonal to a one-parameter family of surfaces Σ, the circles are said to form a cyclic system. In the correspondence established between the surfaces by the circles, the lines of curvature correspond on all the surfaces, and to them correspond also the developables of the congruence formed by the axes of the circles. The planes of the circles envelope a surface S upon which the curves corresponding to the lines of curvature on the surfaces Σ form a conjugate net N, which admits an applicable conjugate net $\overline{N}$. Professor Eisenhart applies to these applicable nets the theorem referred to in his former paper and establishes thereby a transformation of the given cyclic system into other cyclic systems. Corresponding circles of a cyclic system and a transform lie on a sphere, and the nets N and N_1 on the surfaces S and S_1 enveloped by the planes of the circles are in relation T, that is, the developables of the congruence of joins of corresponding points on S and S_1 meet the latter in N and N_1. Moreover, it is possible to establish a correspondence between the surfaces Σ and Σ_1, orthogonal to the two families of circles, so that a surface Σ and the corresponding surface Σ_1 are in the relation of a transformation of Ribaucour, that is, they are the envelope of a two-parameter family of spheres with lines of curvature in correspondence on the two sheets of the envelope.

19. If in a set of differential equations

$$\frac{dx_i}{d\tau} = f_i(\tau, x_1, \cdots, x_n) \qquad (i = 1, \cdots, n)$$

the functions in the second members are considered as variables, the solutions x_i $(i = 1, \cdots, n)$ will be functions, not only of the initial values of τ and the x's, but also of the functions f_i themselves. In Professor Bliss's first paper the properties of these functions of functions are discussed. It is proved, after suitable initial hypotheses have been made, that they are continuous, and that they have what are called in this paper difference functions. It is shown that the possession of a difference function implies also the possession of a differential similar to those which have been defined by Fréchet and Volterra for functions of lines. The theorems of the paper are widely applicable since the variations Δf_i of the functions f_i entering into the differentials are entirely arbitrary. In special cases they may be caused, for example, by variations of parameters in the functions f_i, or, as in ballistics, by variations of other arbitrary functions which occur as arguments in the second members of the differential equations.

20. In Professor Bliss's second paper some functions of lines are considered which arise in ballistics. If $w(y)$ is the velocity of the wind at the altitude y in the direction of the plane of fire of a projectile, then the range X will be a function of the function $w(y)$, and also of the projections x_0', y_0' on the axes of the initial velocity of the projectile. The first differential of this function $X[x_0', y_0', w(y)]$ is the first order effect on the range of wind and of changes in the direction and magnitude of the initial velocity. The paper gives a method of computing the first differential which is of interest theoretically in the theory of functions of lines, and which has proved to be of value in the practical computation of range tables. The situation just described is somewhat simpler than that which arises in practice, for the range is also affected by variations from normal in the density of the air and by the rotation of the earth, and still other disturbing influences will have to be accounted for in the future as the applications of ballistic theory become more exacting. The formulas of the paper are applicable to the cases which have just been mentioned explicitly, and they will undoubtedly also be useful for the computation of other differential corrections which may be necessary in the future. The results obtained are special cases of those of the preceding paper, but they are here deduced independently except for some general theorems which would very likely be accepted by an applied mathematician as intuitively true.

21. Jensen has recently stated, without proof, a theorem regarding the roots of the equations $f'(z) = 0$ and those of $f(z) = 0$, where the coefficients are all real, which locates the

former in many cases more closely than does the Gauss-Lucas "polygon theorem." Professor Curtiss's paper gives a simple proof of Jensen's theorem, and deduces from it the result that at least one pair of roots of $f(z) = 0$ lies in each rectangular hyperbola whose vertices are a conjugate imaginary pair of roots of $f'(z) = 0$. This result is extended so as to give similar theorems regarding the relative positions of roots of $f(z) = 0$ and its derived equations of all orders.

22. It is well known that simplicity and precision are gained by the use of urn schemata in establishing various theorems in the theory of probability. The fundamental importance of urn schemata in mathematical statistics is brought out well by Borel in his Eléments de la Théorie des Probabilités by the statement that the general problem of mathematical statistics is to determine a system of drawings carried out with urns of fixed composition, in such a way that the results of the series of drawings, interpreted by the use of conveniently chosen coefficients, lead, with a very high degree of probability, to a table of values identical with the table of observed values. In the present paper, Professor Rietz presents several urn schemata to show the significance of the correlation coefficient derived from a priori probabilities or most probable frequencies given by pure games of chance. It turns out that in certain cases the correlation coefficient r is simply the ratio of the number of common elements to the total number involved. In other cases, the results, though not so simple, are still capable of interesting interpretations. It is emphasized that such urn schemata may well be made the starting point in the development of the theory of correlation.

23. The theory of the transformations studied by Professor Dines in a paper of the same title in the *Transactions* (volume 20, January, 1919) is further developed in the present paper. The introduction of homogeneous coordinates is attended by the usual advantages. A class of points satisfying a certain type of linear equation is called a lineoid, and serves as an element dual to the point. The question of invariant elements is treated, and provides a means for classifying projective transformations. A theorem of special interest states: A necessary and sufficient condition that a projective trans-

formation be projectively equivalent to an ordinary Fredholm transformation in the same domain is that it admit an invariant lineoid and an invariant point not on that lineoid. In particular every symmetrical projective transformation has this property, but such is not the case with non-symmetrical transformations.

24. The invariants referred to in Professor Shaw's paper are the scalar invariants which appear in the characteristic equation, the expressions previously called *chi*, and related forms. The subject is intimately connected with the Fredholm theory in integral equations.

25. The purpose of the first paper by Professor Carmichael is to suggest a method for deriving a necessary and sufficient condition for the existence of the Stieltjes integral of $f(x)$ as to $u(x)$ from a to b in each of several forms generalizing those frequently employed in the special case of Cauchy-Riemann integration. The theorem of Bliss (*Proceedings of the National Academy of Sciences,* volume 3 (1917), pages 633–637) is included and is proved in a new way.

26. An identity to which he was led by the problem of Stieltjes integration by parts Professor Carmichael applies in the derivation of several theorems relating to the convergence of series. Of these the following is typical: The convergence of the series

$$\sum_{k=1}^{\infty} c_{k-1}^{(2)} c_{k-1}^{(3)} \cdots c_{k-1}^{(n)} (c_k^{(1)} - c_{k-1}^{(1)})$$

is implied by the convergence of the series

$$\sum_{k=1}^{\infty} |c_k^{(i)} - c_{k-1}^{(i)}|, \quad (i = 2, 3, \cdots, n),$$

and the existence of the limits

$$\lim_{k=\infty} c_k^{(1)} c_k^{(2)} \cdots c_k^{(n-1)}, \quad \lim_{k=\infty} c_k^{(1)} \cdots c_k^{(i-1)} c_{k-1}^{(i+1)} \cdots c_{k-1}^{(n)},$$
$$(i = 2, \cdots, n - 1).$$

The instance $n = 2$ of this theorem is classic.

27. In this paper Professor Carmichael throws Stieltjes'

problem of moments into a different form from that of Stieltjes and one readily capable of generalization; and from this he proceeds to a natural physical interpretation of the Stieltjes integral of a bounded function $f(x)$ as to a function $v(x)$ of bounded variation.

28. On page 506 of Merriman and Woodward's Higher Mathematics is given a table showing—for different values of the interpolating factor—the actual and the theoretical average errors of certain interpolated values. Attention is called to the curious fact that when the interpolating factor is an even number the actual error is always less than the theoretical error, while when it is odd the reverse is true, except when the factor is 0.5.

In Professor Howe's paper it is purposed to describe a more exhaustive test of this matter, with special attention to the case where the interpolating factor is 0.5, and also to call attention to an Austrian five-place table from which more accurate results can be obtained than from ordinary tables. Also when only the usual accuracy is sought one can work much more rapidly with this table than with others.

29. In the current volumes of the *Rendiconti* of the Lincei and of the *Seminario* of the University of Rome, Professor Evans expounded a theory of the general equation of Poisson,

$$\int \frac{\partial u}{\partial n} ds = f(e),$$

where $f(e)$ is an arbitrary additive function of point sets, in which he introduced a potential function in the form of a Stieltjes integral. The present report discusses a transformation of this integral into the form involving a polarization vector, and a second form obtained by the slight generalization of an integration by parts of W. H. Young. In the latter case are introduced nuclei which in three dimensions attract according to an inverse fifth power law.

30. In this note Professor Hildebrandt derives a necessary and sufficient condition in order that a sequence of Stieltjes integrals $\int f d\alpha_n$ shall converge to an integral $\int f d\alpha$, for every continuous function on the linear interval $a \leqq x \leqq b$, the functions α_n and α being of bounded variation on the same

interval. He also shows that a linear functional operation on the class of functions of bounded variation on the interval (a, b) is expressible in the form of a Stieltjes integral $\int f d\alpha$, in which f is a continuous function on (a, b); and derives necessary and sufficient conditions that the sequence of Stieltjes integrals $\int f_n d\alpha$ shall converge to an integral of the form $\int f d\alpha$, for every function of bounded variation on the interval (a, b), the functions f_n and f being continuous on the same interval.

31. Professor Roever's paper begins with a statement of the theorems of relative motion concerning the velocities and accelerations of a particle. By applying these theorems to the problem under consideration the equation of motion of a projectile is expressed in vector form in terms of the accelerations due to weight, wind, air resistance and rotation of the earth. From this vector equation the equations of motion are obtained in a form involving the potential function W of the weight field of force, the function F which expresses the retardation due to air resistance and wind, and the angular velocity of the earth's rotation. By replacing W by means of some of its analytic approximations and expressing F in terms of various experimentally determined functions, differential equations of motion of a projectile are obtained which take into consideration air resistance, wind, rotation of the earth, convergence of the verticals, curvature of the layers of constant air density and also other influences, but not the rotation of the projectile due to rifling of the gun.

32. There are known summations of products of the successive coefficients of $(x - y)^k$ by given powers of the numbers of those coefficients, but no results seem to be known for the case of the binomial coefficients, i.e., those of $(x + y)^k$. A recursion formula is given by Professor Cairns for such summations as well as expressions for the leading terms, and an application is made to the evaluation of

$$\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^{2p} dx \quad \text{and} \quad \int_{0}^{\infty} e^{-x^2/2\sigma^2} x^{2p+1} dx.$$

33. Professor Kempner gives a simple method of classification of real polynomial curves with reference to the relative

position and the number of the maxima and minima of the curve, and shows that all types actually exist. For example: for all $\delta > 0$ sufficiently small, the curve

$$y = (x + \delta i)(x - \delta i)(x + 1 + \delta^3 i)(x + 1 - \delta^3 i) \cdot$$
$$(x + 1 + \delta^2 + \delta^5 i)(x + 1 + \delta^2 - \delta^5 i)(x + 1 + \delta^2 + \delta^4)$$

has six extremes which, read for decreasing values of x, are arranged so that the first minimum of y is higher than the second maximum, and the second minimum higher than the third maximum.

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FORM OF THE NUMBER OF SUBGROUPS OF PRIME POWER GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society September 3, 1919.)

§1. *Introduction.*

It is known that the number of the subgroups of order p^α, p being any prime number, which are contained in any group G is always of the form $1 + kp$. When $k = 0$ for every possible pair of values for α and p the group G must be cyclic and vice versa. There are two infinite systems of groups of order p^m containing separately $p + 1$ subgroups of every order which is a proper divisor of the order of the group, viz., the abelian groups of type $(m - 1, 1)$ and the conformal non-abelian groups.

These two infinite systems are composed of all the groups of order p^m involving separately exactly $p + 1$ subgroups of every order which is a proper divisor of p^m. Moreover, if a group of order p^m, $p > 2$, contains exactly $p + 1$ subgroups of each of the two orders p and p^2 it must contain exactly $p + 1$ subgroups of every order which is a proper divisor of the order of the group, and if a group of order 2^m contains exactly three subgroups of each of the orders 2, 4 and 8 it must also contain exactly three subgroups of every other order which is a proper divisor of 2^m.