

THE FEBRUARY MEETING OF THE AMERICAN  
MATHEMATICAL SOCIETY.

THE one hundred and ninety-sixth regular meeting of the Society was held in New York City on Saturday, February 23, extending through the usual morning and afternoon sessions. The attendance included the following seventeen members:

Dr. F. W. Beal, Professor F. N. Cole, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Dr. T. R. Hollcroft, Dr. L. L. Jackson, Mr. S. A. Joffe, Dr. J. R. Kline, Professor E. J. Miles, Professor R. L. Moore, Mr. George Paaswell, Dr. J. F. Ritt, Dr. Caroline E. Seely, Mr. H. E. Webb, Professor H. S. White, Miss E. C. Williams.

Professor H. S. White occupied the chair. The Council announced the election of the following persons to membership in the Society: Miss M. F. Chadbourne, Smith College; Mr. Mervyn Davis, Equitable Life Insurance Company of Iowa; Mr. T. C. Fry, Western Electric Company; Dr. J. E. McAtee, University of Illinois; Dr. Norbert Wiener, Albany, N. Y. Four applications for membership in the Society were received.

The following papers were read at this meeting:

- (1) Dr. J. F. RITT: "Proof of the multiplication formula for determinants by means of linear differential equations."
- (2) Dr. OLIVE C. HAZLETT: "On vector covariants."
- (3) Dr. P. R. RIDER: "On the problem of the calculus of variations in  $n$  dimensions."
- (4) Dr. A. R. SCHWEITZER: "On the iterative properties of an abstract group."
- (5) Dr. A. R. SCHWEITZER: "On certain articles on functional equations."
- (6) Dr. A. R. SCHWEITZER: "On iterative functional equations of the distributive type."
- (7) Dr. J. R. KLINE: "A new proof of a theorem due to Schoenflies."
- (8) Professor R. L. MOORE: "A sufficient condition that a system of arcs should constitute a surface."
- (9) Mr. J. L. WALSH: "On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function."

(10) Professor O. E. GLENN: "Covariant expansion of a modular form."

(11) Dr. J. F. RITT: "Polynomials with a common iterate."

(12) Professor L. P. EISENHART: "Transformations of applicable conjugate nets of curves on surfaces."

(13) Professor S. E. SLOCUM: "The romantic aspect of numbers."

Mr. Walsh's paper was communicated to the Society through Professor Bôcher. In the absence of the authors the papers of Dr. Hazlett, Dr. Rider, Dr. Schweitzer, Mr. Walsh, Professor Glenn, Professor Eisenhart, and Professor Slocum were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Ritt's paper, which is presented only as an interesting piece of manipulation, shows how the formula for the multiplication of determinants can be found from the properties of Wronskian determinants.

2. This paper concerns itself with a certain type of covariant of the general linear algebra which, for convenience, is called a vector covariant. In a previous paper, Dr. Hazlett has developed the theory of a rather different type of covariant, referred to here as scalar covariants, since they involve only the coordinates  $x_i$  of the general number of the algebra  $X = \sum x_i e_i$  and the constants of multiplication  $\gamma_{ijk}$  ( $i, j, k = 1, \dots, n$ ) in contradistinction to the vector covariants which involve the units  $e_i$  ( $i = 1, \dots, n$ ) in addition to the  $x$ 's and  $\gamma$ 's. Since multiplication of the units in general is neither commutative nor associative, one might anticipate some rather inconvenient difficulties. These can, however, be avoided by a device. Accordingly, the annihilators of vector covariants are found and hence it is shown that every rational integral vector covariant of the  $n$ -ary linear algebra is a covariant of the general number  $X = \sum x_i e_i$  of the algebra and a suitable set of scalar covariants of the algebra. From this fact flow theorems analogous to those proved for scalar covariants and and, in particular, the "finiteness" of vector covariants.

3. Dr. Rider's paper considers the theory of minimizing an

integral of the form

$$\int_{t_0}^{t_1} f(x_1, \dots, x_n, \tau_1, \dots, \tau_{n-1}) \sqrt{x_1'^2 + \dots + x_n'^2} dt,$$

in which

$$\tau_i = \arctan x'_{i+1} / \sqrt{x_1'^2 + \dots + x_i'^2}.$$

The Euler equations are derived and the transversality and corner conditions are given, as are the forms of the Hilbert invariant integral and the Weierstrass  $e$ -function. Certain necessary conditions, including the condition of Weierstrass, are obtained. The paper will be published in the *Tôhoku Mathematical Journal*.

4. Dr. Schweitzer gives the following postulates\* for an Abelian group: I<sub>1</sub>.  $x, y$  imply  $\phi(x, y) = z$  uniquely. I<sub>2</sub>.  $x, y$  imply  $f(x, y) = z$  uniquely. II<sub>1</sub>.  $\phi\{x, \phi(y, z)\} = \phi\{\phi(x, y), z\}$ . II<sub>2</sub>.  $\phi\{f(x, y), x\} = x$ . II<sub>3</sub>.  $f\{\phi(x, y), x\} = x$ . II<sub>4</sub>.  $\phi(x, y) = \phi(y, x)$  if  $x \neq y$ . The author shows that if II<sub>4</sub> is replaced by II<sub>4'</sub>,  $\phi(y, y') = \phi(y, y)$  implies  $y = y'$  if  $y \neq y'$ , then the postulates are satisfied by any group, finite or infinite. In the latter set postulate II<sub>1</sub> may be replaced by II<sub>1'</sub>,  $f\{f(y, x), f(z, x)\} = f(y, x)$ , without loss of generality.

In the second part of the paper, on the basis of the preceding postulates, the author points out concepts in group theory expressible in terms of iterative compositions, such as the transform of an element, the commutator of two elements. In particular, infinite sequences of iterative compositional relations are obtained which are satisfied by the elements of any group. Such relations are, e. g., of the type  $\psi\{u_1^{(i)}, u_2^{(i)}, \dots, u_n^{(i)}\} = \psi\{x_1, x_2, \dots, x_n\}$  where  $u_i^{(1)} = \psi\{x_i, t_1, t_2, \dots, t_{n-1}\}$ ,  $u_i^{(2)} = \psi\{t_1, x_i, t_2, \dots, t_{n-1}\}$  etc., and  $i = 1, 2, \dots, n$  ( $n = 2, 3, \dots$ ).

The preceding postulates are free from explicit statements of existence. In the *Transactions* (1905) Dickson in effect eliminates the explicit existential properties of the Moore-Dickson postulates for a finite group.

5. Dr. Schweitzer's second paper is a constructive criticism of certain articles† on functional equations due to Suto on the

\* Cf. Grassmann, *Gesammelte Werke*, vol. I<sub>2</sub>, p. 239.

† *Tôhoku Math. Journal*, vol. 3 (1913), vol. 6 (1914-1915).

basis of the author's independent investigations. Numerous errors of Suto are pointed out; in particular the methods by which the iterative functional equation, loc. cit., volume 3, page 60, is solved are erroneous. This equation is but one of an important class of iterative equations previously defined by the author, e. g.,

$$f\{u_1, u_2, \dots, u_n\} = f\{v_1, v_2, \dots, v_n\} \quad (n = 2, 3, \dots),$$

where

$$u_i = f\{x_{1i}, x_{2i}, \dots, x_{ni}\}, v_i = f\{x_{j_1k_i}, x_{j_2k_i}, \dots, x_{j_nk_i}\}$$

and  $x_{j_1k_i}$  is some one of the arguments of  $u_1$ ,  $x_{j_2k_i}$  is some one of the arguments of  $u_2$ , etc., and  $i = 1, 2, \dots, n$ . The equation treated by Suto was solved by the author by direct differentiation. Concerning the "associative" functional equations the author obtains a number of theorems the simplest of which is: If  $\phi\{x, \phi(y, z)\} = \phi\{\phi(x, y), z\}$  and  $\phi(x', y) = z$  implies  $x' = f(z, y)$  then  $\phi(x, y) = \psi^{-1}\{\psi(x) + \psi(y)\}$  and  $f(x, y) = \psi^{-1}\{\psi(x) - \psi(y)\}$ . The equation of Suto, loc. cit., volume 6, page 82, is a special case of the equation

$$\sum_{i=1}^n f_i(x, y) = \psi(x, y),$$

where the functions  $f_i(x, y)$  satisfy given partial differential equations of the first order and  $\psi(x, y)$  is a given "quasi-transitive" or "symmetric" function, e. g.,  $x - y, x/y, x + y, x \cdot y$ . In particular important functional equations arise if the  $f_i(x, y)$  are required to satisfy given iterative functional equations. Finally Suto's paper on the arithmetic mean, loc. cit., volume 6 (page 79), is compared with the author's.

6. Dr. Schweitzer proves the following theorem: A sufficient condition that

$$(1) \quad \phi\{\phi(x_{11}, \dots, x_{1n}), t_1, t_2, \dots, t_{n-1}\} \\ = \phi\{\phi(x_{11}, t_1, \dots, t_{n-1}), \dots, \phi(x_{1n}, t_1, \dots, t_{n-1})\},$$

where  $n = 2, 3, \dots$ , is that

$$(2) \quad \phi\{\phi(x_{11}, \dots, x_{1n}), \dots, \phi(x_{n1}, \dots, x_{nn})\} \\ = \phi\{\phi(x_{11}, \dots, x_{n1}), \dots, \phi(x_{1n}, \dots, x_{nn})\}$$

and

$$\begin{aligned} \phi\{\phi(x_{11}, \dots, x_{1n}), \phi(t_1, \dots, t_1), \dots, \phi(t_{n-1}, \dots, t_{n-1})\} \\ = \phi\{\phi(x_{11}, \dots, x_{1n}), t_1, t_2, \dots, t_{n-1}\} \end{aligned}$$

and consequently infers a solutions of (1). The result is generalized to iterative equations of infinite degree.\* The equation (1) is a distributive equation of order 1; equation (2) is a limiting case, viz., of order  $n$ . Equations of order  $k$ ,  $1 \leq k \leq n$  are readily defined, e. g.,

$$\phi\{\phi(x_1, x_2, x_3)\phi(y_1, y_2, y_3)t\} = \phi\{\phi(x_1, y_1, t)\phi(x_2, x_2, t)\phi(x_3, y_3, t)\}$$

is distributive of the second order. Extension to distributive iterative equations on two functions is made.

7. In a paper, presented to the Society at the December meeting, Professor R. L. Moore proved the following theorem: If the points  $A$  and  $C$  separate  $B$  and  $D$  on the simple closed curve  $J$ , then it is possible to draw a set of simple continuous arcs such that through every point of  $J$  and its interior there are two and only two arcs of the set, the arcs  $AB$ ,  $BC$ ,  $CD$  and  $DA$  being arcs of the set. This theorem was proved on the basis of his set of axioms  $\Sigma_1$ †. With the use of this theorem Dr. Kline gives a simple proof of the following theorem, due to Schoenflies: Given two simple closed plane curves  $J_1$  and  $J_2$ , which are in one-to-one continuous correspondence under a correspondence  $\Pi$ . Let  $R_i$  ( $i = 1, 2$ ) denote the point set composed of  $J_i$  plus  $I_i$ , the interior of  $J_i$ . Then there exists a continuous one-to-one correspondence between the points of  $R_1$  and  $R_2$  such that the points of  $J_1$  and  $J_2$  correspond as fixed by the correspondence  $\Pi$ . For an abstract of a proof of this theorem by Dr. G. A. Pfeiffer from a somewhat different point of view see this BULLETIN, volume 24, number 4, page 171.

8. Suppose that  $AB$  and  $CD$  are two arcs in three-dimensional space. Suppose that  $G$  is a bounded self-compact system of simple continuous arcs such that each arc of  $G$  has its end points on  $AB$  and  $CD$  respectively and no two arcs of

\* Cf. Suto, loc. cit., vol. 6 (1914-1915), p. 16.

† Cf. R. L. Moore, "On the foundations of plane analysis situs," *Transactions Amer. Math. Society*, volume 17 (1916), pp. 131-164.

$G$  have any point in common. Professor Moore proposes to show: (1) that the point set constituted by such a system of arcs cannot always be brought into continuous one-to-one correspondence with the surface consisting of a circle plus its interior; (2) that such a correspondence will exist in case the arcs of the system  $G$  are equicontinuous. He has under consideration further theorems of this type.

9. Mr. Walsh's paper will be published in the *Transactions*.

10. Professor Glenn's paper contains elaboration, and computations based on particular cases, of a former theorem of his concerning the reducibility (mod  $p$ ) of the general binary quantic. The forms of orders  $> p^2 - 1$  ( $p = 2, 3$ ) are reducible modulo  $p$  in terms of modular covariants of orders 0 to  $p^2 - 1$ , as polynomials or covariant expansions in the universal covariants. Such an expansion is a form of generalized modular typical representation in which the multiplier concomitant is unity. Tables are given showing the expansions and all constituent covariants occurring in them, for forms of the first eleven orders modulo 2, and for the orders 9, 10, 11 when the modulus is 3.

The paper appeared in the March, 1918, number of the *Annals of Mathematics*.

11. Dr. Ritt determines the conditions under which two different polynomials  $F(x)$  and  $f(x)$ , both of degree  $n$  greater than one, can have a common  $p$ th iterate.  $F(x)$  can be derived by means of a linear integral transformation from a polynomial  $g(x) = x^n + k_1x^{n-q} + k_2x^{n-2q} + \dots + k_r x^{n-rq}$ , where  $q$  is some divisor of  $(n^p - 1)/(n - 1)$ , and  $f(x)$  will be derived by the same transformation from  $\omega g(x)$ , where  $\omega$  is a  $q$ th root of unity. The proof is accomplished with the help of an important theorem on iteration due to L. Boettcher, which has lain hidden for a long time in a paper published in Russian.\*

12. When two surfaces are applicable to one another, in the correspondence thus established there is a unique conjugate system of curves on each surface corresponding to a conjugate system on the other. These two conjugate systems, or nets, are said to be applicable nets. Let  $N$  and  $\bar{N}$  be two such applicable nets. Peterson showed that when a net  $N'$ , parallel

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\* *Bulletin of the Kusan Math. Society*, vol. 14 (1905) p. 176.

to  $N$ , is known, a net  $\bar{N}'$ , parallel to  $\bar{N}$ , can be found by quadratures such that  $N'$  and  $\bar{N}'$  are applicable. Professor Eisenhart has applied to applicable nets the theory of transformations  $T$  (cf. *Transactions*, volume 17, page 97), and has established transformations of applicable nets into applicable nets. The transformations of nets admitting an infinity of applicable nets into similar nets are of particular interest. The paper will be published in the April number of the *Transactions*.

13. Professor Slocum's paper calls attention to the important rôle played by number symbolism in the intellectual development of mankind. Wherever mind has reacted to the stimulus of natural phenomena, the number concept has resulted as the inevitable expression of the laws governing the material universe. The properties of number, first accepted as a fact, subsequently came to be regarded as symbolic, and only in recent times has the mind been able to grasp their true significance as one aspect of the great principle of functionality.

Various instances are cited showing that the ancient customs of all races imply a distinct belief in the action of numbers on the course of human events. Beginning with the empiricism of primitive nations, like the Egyptians, Persians, and Greeks, it is shown that as civilization developed number ideas became symbolic, and the properties of numbers were invested with physical attributes. Numerals were regarded not merely as passive symbols but as active principles of good and evil, capable of reproducing their properties in accordance with mathematical operations. The extensive use of number symbolism by the ancient Hebrews, however, stands in marked contrast to that characteristic of pagan nations like the Greeks. With the Hebrews number symbolism was simply incidental to their rich Oriental imagery, and there is no evidence anywhere in Old Testament literature that the properties of numbers were even known, much less regarded as exerting any mystic influence on human destiny. But when the New Testament was written the Hebrews had outgrown their isolation, and Grecian influence is clearly apparent in the mystic character given to number symbolism, especially in the Apocalypse.

During the middle ages the attempt of scholasticism to reconcile the tenets of Christianity with Grecian philosophy

lent special impetus to the study and development of sacred hermeneutics. The growth of this peculiar by-product of scholasticism is outlined, with references to the work on number symbolism of the Venerable Bede, St. Augustine, and numerous other prominent ecclesiastics. In particular, a remarkable example is given of a ninth century rendering of the account given in Genesis 18 of the conflict of the 318 servants of Abraham against the four kings. The classic example of the number of the beast is also studied, and it is shown that its interpretation serves to establish the date of the writing of Revelation, long in dispute, as well as affords a key to the system of number symbolism used by St. John.

F. N. COLE,  
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## SOME REMARKABLE DETERMINANTS OF INTEGERS.

BY PROFESSOR E. T. BELL.

1. THE determinants in this note are arithmetical rather than algebraic in character; their properties, not obvious by the usual reductions, follow immediately from simple considerations in the theory of numbers. Throughout, letters other than  $x$  and functional signs denote positive integers, and  $[x]$  is the greatest integer in  $x$ .

2. Let  $D_n | F(k), G(k) |$  denote the determinant of the  $n$ th order whose first and last columns are respectively  $F(1), F(2), \dots, F(n)$ , and  $G(1), G(2), \dots, G(n)$ ; and whose  $(1+k)$ th column ( $k = 1, 2, \dots, n-2$ ) is derived from the first by prefixing  $k$  zeros and repeating in succession each element of the first  $(1+k)$  times, until in all a column of  $n$  elements has been written down

$$(1) \quad D_n | F(k), G(k) | \equiv \begin{vmatrix} F(1) & 0 & 0 & \cdots & G(1) \\ F(2) & F(1) & 0 & \cdots & G(2) \\ F(3) & F(1) & F(1) & \cdots & G(3) \\ F(4) & F(2) & F(1) & \cdots & G(4) \\ F(5) & F(2) & F(1) & \cdots & G(5) \\ F(6) & F(3) & F(2) & \cdots & G(6) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$