does not exist, and, therefore, are steps towards the solution of the hitherto so-called Steiner problem (*a*).

The school-girl problem is merely an example which originated in the development of this paper on combinations, and Kirkman justly complained of the almost total eclipse of this paper in the wide popular interest aroused by the school-girl problem. The eclipse appears to have continued up to the present day, since no mention is made of this Kirkman paper by Steiner, Reiss, Netto, or by any of the recent writers on triad systems.

VASSAR COLLEGE,
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PIERRE LAURENT WANTZEL.

BY PROFESSOR FLORIAN CAJORI.

(Read before the American Mathematical Society September 4, 1917.)

EVERY one knows that one of the noted proofs of the impossibility of an algebraic solution of the general quintic equation is due to Wantzel. Nevertheless histories of mathematics and biographical dictionaries are silent regarding his life. The eleven papers listed in Poggendorff's Handwörterbuch as due to "Pierre Laurent Wantzell" do not include the proof in question, and a query is raised in a footnote regarding another "Wantzell"; but nowhere does Poggendorff refer to a "Wantzel." Text-books on algebra and the theory of equations do not give Wantzel's full name. The reader is thus left without positive information as to the author of "Wantzel's proof." His name suggests German nationality, as does the name of "Mannheim," of slide-rule fame. Yet both these men were born in Paris and passed their lives at the Polytechnic School there.* Born in 1814, Wantzel died prematurely in 1848. He is the "Pierre Laurent Wantzell" of Poggendorff but in his published articles his name is always spelled

* On the life of Wantzel, see Barré de Saint-Venant in *Nouvelles Annales de Mathématiques* (Terquem et Gerono), vol. 7 (1848), pp. 321–331; A. de Lapparent in Ecole polytechnique, Livre du Centenaire, 1794–1894, vol. I., Paris, 1895, pp. 133–135, see also pp. 63–65, 190; Gaston Pinet's Ecrivains et Penseurs Polytechniciens, 2e éd., Paris, 1902, p. 20; Charles Sturm in *Comptes rendus hebdomadaires des Séances de l'Académie des Sciences*, Paris, vol. 28 (1849), pp. 66, 67.

"Wantzel." Pinet says of him: "Endowed with extreme vivacity of impressions and with truly universal aptitudes, he carried off the prize for a French dissertation and a Latin dissertation at a general competition and, the year following (1832), entered with first rank the Polytechnic School—a double success before unheard of. He studied with inveterate zeal German and Scotch philosophers; he threw himself into mathematics, philosophy, history, music, and into controversy, exhibiting everywhere equal superiority of mind." He became élève-ingénieur des Ponts et Chaussées, then ingénieur; he was appointed répétiteur about the time when such positions were held by Comte, Transon, Bertrand, Bonnet, Catalan, Leverrier, and Delaunay. In 1843 he became examinateur d'admission. Saint-Venant says of him: "He was blameworthy for having been too rebellious to the counsels of prudence and of friendship. Ordinarily he worked evenings, not lying down until late; then he read, and took only a few hours of troubled sleep, making alternately wrong use of coffee and opium, and taking his meals at irregular hours until he was married. He put unlimited trust in his constitution, very strong by nature, which he taunted at pleasure by all sorts of abuse. He brought sadness to those who mourn his premature death."

The Royal Society Catalogue of Scientific Papers quotes the titles of 18 papers by Wantzel, and of three more which he brought out jointly with Saint-Venant.

General Quintic Insolvable by Radicals.

As previously stated, Wantzel's most noted scientific achievement is found in his paper "De l'impossibilité de résoudre toutes les équations algébriques avec des radicaux" in the *Nouvelles Annales de Mathématiques*, volume 4 (1845), pages 57–65. The second part of this proof, which involves substitution theory, is reproduced in Serret's Algèbre supérieure.*

At the beginning of his article, Wantzel expresses himself on the proofs of Abel and Ruffini as follows: "Although his (Abel's) demonstration is at bottom exact, it is presented in a form too complicated and so vague that it is not generally accepted. Many years previous, Ruffini, an Italian geometer,

* In the fifth edition, 1885, the part proof occurs in vol. II, pp. 512–517.

had treated the same question in a manner much vaguer still and with insufficient developments, although he had returned to this subject many times. In meditating on the researches of these two geometers and with the aid of principles which we established before,* we have arrived at a form of proof which appears so strict as to remove all doubt on this important part of the theory of equations." His previous paper, to which reference is made, deals with algebraic incommensurables. Wantzel says that the main theorem set forth therein had been previously established by Abel and again by Liouville. Thus, Wantzel expresses special indebtedness to Abel and Liouville, while to Ruffini he makes acknowledgment only in a general way as indicated in the passage quoted above. A different estimate of indebtedness has been made more recently by H. Burkhardt† and E. Bortolotti.‡ They claim that the proof which goes by the name of Wantzel is essentially Ruffini's proof of 1813. Inasmuch as Wantzel's proof has been generally accepted as altogether valid, the implication is that Ruffini's proof is equally valid.

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* *Nouvelles Annales* (Terquem), vol. 2 (1843), pp. 117–127.

† H. Burkhardt, "Die Anfänge der Gruppentheorie und Paolo Ruffini," *Zeitsch. f. Mathematik u. Physik*, 37. Jahrg., Supplement, Leipzig, 1892, pp. 119–159. Burkhardt says (p. 156): "Es braucht wohl kaum noch ausdrücklich hervorgehoben zu werden, dass diese Fassung des Unauflösbarkeitsbeweises sich in allen wesentlichen Punkten mit derjenigen deckt, welche als 'Wantzel'sche Modification des Abel'schen Beweises' in den Lehrbüchern mitgeteilt zu werden pflegt." Again he says (p. 159), "diesen Beweis hat er (Ruffini) nicht nur zuerst durchgeführt, sondern ihn auch nach verschiedenen Umarbeitungen auf die einfache Form gebracht, welche Wantzel zugeschrieben zu werden pflegt."

‡ In the Carteggio di Paolo Ruffini con alcuni scienziati del suo tempo, Roma, 1906, p. 15 (303), Bortolotti says in a footnote that Ruffini's Riflessioni intorno alla soluzione delle equazioni algebraiche generali, Modena, 1813, "contiene la dimostrazione, che per lungo tempo attribuita al Wantzel, fu sempre considerata come la più facile e la più convincente del teorema di Ruffini." Bortolotti expresses himself more fully in his discourse, Influenza dell' Opera Matematica di Paolo Ruffini (Modena), 1902, p. 42: "E sarebbe anche strano che egli (Abel) accusasse la dimostrazione di Ruffini di esser troppo complicata; laddove essa è tanto semplice che, quando il *Wantzel* volle ridurre il teorema a forma facile e piana, fu costretto a riprodurre, nelle sue linee generali, la redazione del Ruffini. L' ingiustizia delle umane cose ha però voluto, e vuole tuttora, che la dimostrazione che il Ruffini ha data del teorema da lui stesso scoperto, sia stata battezzata col nome di *Dimostrazione di Wantzel del Teorema di Abel.* Quella disgraziata denominazione, introdotta dal Serret nel suo rinomato libro di *Algèbre supérieure* è stata ricopiata di testo in testo fino ai nostri giorni: e si trova anche nell' ultimo e più reputato testo di Analisi, quello del Picard, uscito nel 1896 quattro anni dopo la piena ed irrefutabile dimostrazione data dal Burkhardt della priorità del Ruffini."

The question arises, is Wantzel's proof really the same as Ruffini's of 1813? Before answering this query, the question must be considered, how much must be included as constituting an integral part of the proof? Is only the portion involving substitution theory to be taken into account, or is the introductory part relating to irrational algebraic functions and the general expression for the root form of an algebraically solvable equation to be considered as well? Burkhardt and Bortolotti seemingly look upon the application of substitution theory as the only essential part of the demonstration. But, as Netto says, the "proof of the impossibility of an algebraic solution of general equations above the fourth degree can never be obtained from the theory of substitutions alone." *

Moreover the state of algebra at the beginning of the nineteenth century required that both parts be investigated, to establish the impossibility. With Ruffini and Abel, and also with Wantzel, the establishment of the general root form of algebraically solvable equations was the most difficult and serious part which occupied the larger portion of the space devoted to the proofs. Both parts received consideration by William Rowan Hamilton,† L. Kronecker,‡ O. Hölder,§ and J. Pierpont.‖ Sylow¶ and Hölder point out that there is an unproved assumption in the *first part* of Ruffini's proof. Neither Ruffini nor Abel were ever seriously criticized for lack of rigor in the strictly substitution theory part, but both were criticized on the other parts. Abel committed an error in the statement of the theorem relating to an algebraic function of the μth order and mth degree, which was corrected by L. Königsberger in 1869.** This error had disturbed William Rowan Hamilton, who, in his long article†† "On the argument of Abel," declared that the error "renders it difficult to judge of the validity of his (Abel's) subsequent reasoning." However, J. Pierpont‡‡ rightly remarks that this slip "does not

* E. Netto, Theory of Substitutions, translated by F. N. Cole, Ann Arbor, 1892, p. 240.

† W. R. Hamilton, "On the argument of Abel," *Transactions of the Royal Irish Society of the year 1839*, p. 248.

‡ L. Kronecker, *Monatsbericht d. k. p. Akademie d. Wiss. zu Berlin*, 1879, p. 205.

§ Encyklopädie d. math. Wissensch., 1. Band, p. 504.

‖ J. Pierpont in *Monatshefte f. Math. u. Physik*, VI (1895), pp. 37–51.

¶ Oeuvres of Abel, edition by Sylow and Lie, II, 1881, p. 293.

** L. Königsberger, *Math. Annalen*, vol. 1 (1869), pp. 168, 169.

†† W. R. Hamilton, loc. cit., p. 248.

‡‡ J. Pierpont, loc. cit., p. 47.

affect in the least the validity of the proof." Abel establishes the fundamental theorem, first enunciated by Vandermonde: "If an equation is algebraically solvable, then one can always give its root such a form that all algebraic functions of which it is composed, can be expressed by rational functions of the roots of the given equation." This theorem is fundamental also in Ruffini's research but he failed to give it convincing proof. It is to Abel's credit to have given it a binding demonstration. Abel gives a classification of algebraic functions involving radicals. Wantzel follows Abel, but gives a shorter treatment containing only what is needed for his immediate purpose. In his paper of 1843, Wantzel deals with the properties and classification of radicals and of rational functions of radicals that involve root extractions of real but not of imaginary "quantities." In his paper of 1845, in which the subject is continued and the Ruffini-Abel theorem is established, he states that the results previously proved for radicaux numériques apply equally to radicaux algébriques and proceeds on this assumption. Wantzel's treatment of this first part of his proof is not always happy; Serret in his Algèbre supérieure prefers to adhere to Abel's exposition. There is nothing in Ruffini that directly suggested to Wantzel his mode of classifying algebraic functions. On the other hand, Wantzel's indebtedness to Abel is strongly evident. Nor does Wantzel attempt to conceal this fact. Besides the general statement of indebtedness to which we referred earlier, Wantzel makes special reference to Abel in two places: one where he says that his names for the different classes of radicals are pretty nearly the same as Abel's; the other where he states that his fundamental theorem on radicals was proved before him by Abel and Liouville.*

The Ruffini publication of 1813, previously referred to, consists of viii + 140 pages. The impossibility of an algebraic solution of the general equation of a degree higher than the fourth is given in Part I; Part II treats of the impossibility of solution by the aid of certain transcendental expressions. Here again, a comparatively small portion concerns itself with the part of the proof involving substitution theory. The larger part of the book is devoted to the study of the properties of algebraic functions and of a certain class of tran-

* See Wantzel in *Nouvelles Annales de Mathématiques*, Tome II, Paris, 1843, pp. 125, 127.

scendental functions (espressione trascendente esatta) which, among others, includes trigonometric and logarithmic functions, but is not sharply defined; nor does the many-valuedness of these transcendental functions receive adequate attention. The proof of the impossibility of solution by an espressione trascendente esatta possesses therefore little value.

If, for the moment, we leave out of consideration everything that Ruffini says about transcendental functions, and confine ourselves to his proof of the impossibility of an *algebraic* solution, we find that, although inconclusive in some important details, it must be admitted to his everlasting credit that the general outline of his demonstration is correct. The part of the proof regarding the form which an algebraic root must take (if such an algebraic root exist) is the weak part. Certain theorems necessary for his proof are not established, or rest upon illusory arguments. Letting P be a rational function of the coefficients of the given equation, he introduces the first radical Q by the relation $Q^p = P$ (p prime), then he lets F_1 be a rational function of P and Q, and defines the second radical R by the relation $R^q = F_1$ (q prime), etc. His general root form is $x_n = F$ (P, Q, R, S, etc.), where (page 12) F is a rational function of $P, Q, R, S, \cdots$. It is *assumed* by Ruffini that if the equation can be solved by algebra, the form of its roots must be the one given here.

The part of Ruffini's proof involving substitution theory is free of fault and is in outline as follows:

(1) Let P be a symmetric function of the roots $x_1, x_2, \cdots, x_5$ and unaltered by the cyclic substitution $s = (12345)$. Applying $s, s^2, \cdots$ to y_1, where $y_1^p = P$, gives $y_1^p = y_2^p = \cdots = y_5^p$, and $y_2 = \beta y_1, \cdots, y_5 = \beta^4 y_1, y_1 = \beta^5 y_1$; hence $\beta^5 = 1$.

(2) P is unaltered by $(123) = \sigma$. Applying $\sigma, \sigma^2, \cdots$ to y_1, we get by the above process $y_a = \gamma y_1, y_{a+1} = \gamma^2 y_1, y_1 = \gamma^3 y_1$ and $\gamma^3 = 1$,

(3) The substitutions $(12345)(123) = \tau, \tau^2, \cdots$ applied to y_1 give similarly $\beta^5\gamma^5 = 1$. Hence $\gamma = 1$, and $y_1 = y_a = y_{a+1}$.

(4) The substitutions $(345) = \phi, \phi^2, \cdots$ applied to y_1 give similarly $y_c = \delta y_1, y_{c+1} = \delta^2 y_1, y_1 = \delta^3 y_1$. Hence $\delta^3 = 1$.

(5) The substitutions $(12345)(345) = \pi, \pi^2, \cdots$ applied to y_1 yield $\beta^5\delta^5 = 1$, hence $\delta = 1$ and $y_1 = y_c = y_{c+1}$. Hence $y_1 = y_a = y_c$. Applying σ to these yields $y_1 = y_a = y_2$, hence $\beta = 1$ and Q is unaltered by $s, s^2, \cdots$. The same conclusion can be reached for the algebraic irrationals $R, S, \cdots$, and

finally for F. Hence $x_n = F(P, Q, R, \cdots)$ is absurd, since its right member is unaltered by certain substitutions while the left member is altered.

The part of Wantzel's proof involving substitution theory is as follows:

(1) Let p be a symmetric function of the roots $x_1, x_2, \cdots x_5$, and y a rational function of the roots, and $y^n = p$. The transposition (12) yields $y_1 = \alpha y$, $y = \alpha y_1$; hence $\alpha^2 = 1$ and $n = 2$; the first radical appearing in the root form is a square root.

(2) In $z^n = p_1$ let z be a rational function of the roots which is altered by a cyclic substitution of three letters, but p_1 is unaltered by it. We obtain $z_1 = \alpha z$, $z_2 = \alpha z_1$, $z = \alpha z_2$; hence $\alpha^3 = 1$ and $n = 3$.

(3) Applying to this same z a cyclic substitution of five letters, he proves that $\alpha^5 = 1$. But $\alpha^3 = 1$; hence $\alpha = 1$.

(4) Hence when the degree of the equation exceeds four, z is invariant under a cyclic substitution of three letters. The relation between the roots $x_1 = \psi(x_1, x_2, x_3, x_4, x_5)$ must be an identical equation. This is impossible since the left member is altered by (123) and the right member is not.

The substitution theory as applied by Wantzel is reminiscent of that of Ruffini, yet is far from identical with it. Wantzel stands no closer to Ruffini in this second part of the proof than he stands to Abel in the first part. It is what one might have expected Wantzel to contribute, after "meditating on the researches of the two geometers." The proofs of Ruffini and Wantzel differ altogether in the first part. Hence the claim put forth by Burkhardt and Bortolotti, that Wantzel's proof is the same as Ruffini's, is wholly unsupported by the facts as regards the first part and is too sweeping as regards the second part.

The "Irreducible Case" in Cubics.

Quite forgotten are the proofs given by Wantzel of three other theorems of note, viz., the impossibility of trisecting angles, of duplicating cubes, and of avoiding the "irreducible case" in the algebraic solution of irreducible cubics. For these theorems Wantzel appears to have been the first to advance rigorous proofs. To be sure, Paolo Ruffini, in his booklet of 1813 (pages 54–57), had put forth a proof that the "irreducible case" is unavoidable, but in the absence of satisfactory demonstration that the root must take a certain form, it is open to objection. Wantzel proves this theorem at the end of his paper of 1843, previously referred to. He writes a root of the irreducible cubic in the form of an entire radical function of the nth kind (espèce), $x = A + Bu + \cdots + Mu^{m-1}$, where $u = \sqrt[m]{a}$, a is a radical function of the $(n - 1)$th kind, while $A, B, \cdots, M$ may be such functions of the nth kind but are of a degree inferior to x. Roots of imaginary numbers do not occur in x. He shows that B may be taken ± 1. Substituting for u in x the roots of the irreducible equation $u^m - a = 0$, he finds that $m = 3$ or 2. If $m = 3$ and $\alpha^3 = 1$, he proves that

$$x + \alpha x_1 + \alpha^2 x_2 = \pm 3u.$$

Since, by supposition, x, x_1, x_2, u are all real, it follows that $x_1 = x_2$, which is impossible since the given cubic is irreducible. He shows that $m = 2$ is likewise impossible. Hence the "irreducible case" cannot be avoided. Nearly half a century later proofs were given of this theorem by Mollame (1890), Hölder (1891), and Kneser (1892).

Duplication of Cubes and Trisection of Angles.

These problems are taken up in Wantzel's article in *Liouville's Journal*, volume 2 (1837), pages 366–372, on the "means of ascertaining whether a geometric problem can be solved with ruler and compasses." He shows first that problems solvable by ruler and compasses can be solved algebraically by a series of quadratic equations, (S) $x_i^2 + A_{i-1}x_i + B_{i-1} = 0$ $(i = 1, 2, \cdots, n)$ where A_0 and B_0 are rational functions of given numbers $p, q, \cdots$, and where A_i and B_i are rational functions of $x_i, x_{i-1}, \cdots, x_1, p, q, \cdots$. Secondly, he shows that, if in A_{n-1} and B_{n-1} we substitute in succession the two values of x_{n-1} obtained by solving $x_{n-1}^2 + A_{n-2}x_{n-1} + B_{n-2} = 0$, and if thereupon we multiply together the two resulting expressions for the left member of $x_n^2 + A_{n-1}x_n + B_{n-1} = 0$, we obtain an equation of the fourth degree in x_n. Its coefficients are rational functions of $x_{n-2}, x_{n-3}, \cdots, x_1, p, q, \cdots$. Repeating this process by eliminating successively $x_{n-2}, \cdots, x_1$, he obtains one equation of the degree 2^n in x_n, the coefficients of which are rational functions of $p, q, \cdots$. Thirdly, he shows that an equation of the degree 2^n, resulting from the least possible number of quadratic equations necessary to solve a given problem by ruler and compasses, is irreducible. Fourthly, he considers tests for determining whether a proposed irreducible equation of the degree 2^n can be solved by a series of square root extractions. This is done by equating the coefficients of the proposed equation with those of the general equation of the degree 2^n obtained from the system (S). If thereby the coefficients of the quadratic equations (S) can be obtained by the extraction of square roots, but of no higher roots, then the proposed equation can be solved in that manner.

Wantzel then remarks that the equation $x^3 - 2a^3 = 0$, arising in the problem of the duplication of a cube, is irreducible, but is not of the degree 2^n; hence the cube cannot be doubled in volume by a construction with ruler and compasses. He draws the same conclusion for the equation $x^3 - \frac{3}{4}x + \frac{1}{4}a = 0$, on which depends the trisection of an angle. His own words are: "Cette équation est irréductible si elle n'a pas de racine qui soit une fonction rationnelle de a et c'est ce qui arrive tant que a reste algébrique; ainsi le problème ne peut être résolu en général avec la règle et le compas. Il nous semble qu'il n'avait pas encore été démontré rigoureusement que ces problèmes, si célèbres chez les anciens, ne fussent pas susceptibles d'une solution par les constructions géométriques auxquelles ils s'attachaient particulièrement."

Saint-Venant admits Wantzel's claim of priority and adds that somewhat later Charles Sturm simplified the proofs but did not publish them. So far as now known, Wantzel's priority in publishing detailed, explicit and full proofs of the impossibility of doubling cubes, of trisecting angles and of avoiding the "irreducible case" in the cubic is incontested.

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