THE WINTER MEETING OF THE CHICAGO SECTION.

THE fortieth meeting of the Chicago Section was held at the University of Chicago on December 28–29 in conjunction with the third annual meeting of the Mathematical Association of America. The meeting opened with a dinner of the two organizations on Thursday evening, December 27, at which seventy-three persons were present. The program for the joint session on Friday afternoon consisted of the retiring address by the chairman of the section, Professor W. B. Ford, entitled "A conspectus of the modern theory of divergent series," and of a paper by Professor L. D. Ames, "On a definition of the real number system by means of infinite decimals."

The attendance at the different sessions, which continued until Saturday afternoon, included about eighty-five persons, among whom were the following fifty-nine members of the Society:

Professor L. D. Ames, Professor R. C. Archibald, Professor G. A. Bliss, Professor P. P. Boyd, Professor H. T. Burgess, Professor W. H. Bussey, Professor W. D. Cairns, Professor Florian Cajori, Professor D. F. Campbell, Professor R. D. Carmichael, Professor C. E. Comstock, Professor A. R. Crathorne, Mr. G. H. Cresse, Professor D. R. Curtiss, Professor E. W. Davis, Professor L. E. Dickson, Professor E. L. Dodd, Professor Arnold Dresden, Professor Otto Dunkel, Professor Arnold Emch, Professor L. C. Emmons, Professor H. J. Ettlinger, Professor W. B. Ford, Professor Harris Hancock, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor F. H. Hodge, Professor E. V. Huntington, Professor A. M. Kenyon, Dr. H. R. Kingston, Professor Kurt Laves, Dr. E. B. Lytle, Professor Malcolm McNeill, Professor W. D. MacMillan, Professor G. A. Miller, Professor U. G. Mitchell, Professor C. N. Moore, Professor E. H. Moore, Professor C. C. Morris, Professor E. J. Moulton, Professor G. W. Myers, Professor H. B. Phillips, Professor L. C. Plant, Professor S. E. Rasor, Professor H. L. Rietz, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Dr. G. W. Smith, Professor E. J. Townsend, Professor H. W. Tyler, Professor J. N. Van der Vries, Professor L. G. Weld, Professor E. J. Wilczynski, Professor A. E. Young, Professor J. W. A. Young.

At the business meeting on Saturday forenoon the following officers were elected: chairman, Professor G. A. Bliss; secretary, Professor Arnold Dresden; third member of programme committee, Dr. A. J. Kempner. The section decided to elect its officers for terms of two years, the new practice to start with the officers elected at this meeting. A committee was appointed to consider the advisability of holding a symposium at the next spring meeting and to make the necessary arrangements in case the decision is favorable.

The sessions of Friday forenoon and Saturday forenoon were presided over by Professor Ford, chairman of the Section; the joint session of Friday afternoon by the president of the Society, Professor L. E. Dickson, and the session of Saturday afternoon by Professor G. A. Bliss, chairman-elect of the Section.

The following papers were presented at this meeting:

(1) Dr. G. E. Wahlin: "A study of the principal units of an algebraic domain $\mathfrak{k}(\mathfrak{p}, \mathfrak{a})$."

(2) Dr. C. H. Forsyth: "Supernormal curves."

(3) Professor E. L. Dodd: "A comparison of certain functions of measurement, assuming the Gaussian probability law which provides for a constant error."

(4) Dr. E. W. Chittenden: "On the limit functions of sequences of continuous functions converging relatively uniformly."

(5) Dr. A. R. Schweitzer: "On the iterative functional compositions of index $(n + 1, k)$, $(n, k = 1, 2, \cdots)$ and associated functional equations."

(6) Dr. A. R. Schweitzer: "On a description of mathematics."

(7) Professor Arnold Emch: "On the theory of isotropic curves."

(8) Professor H. B. Phillips: "The relation of cubical determinants to the double multiplication of Gibbs."

(9) Professors H. B. Phillips and C. L. E. Moore: "The dyadics occurring in a projective point space of three dimensions."

(10) Professor R. D. Carmichael: "On sequences of integers defined by recurrence relations."

(11) Professor C. N. MOORE: "On the convergence of the developments in Bessel's functions."

(12) Professor G. A. MILLER: "Sets of independent generators of a substitution group."

(13) Professor J. B. SHAW: "Infinite linear associative algebras."

(14) Professor H. T. BURGESS: "Note on the reduction of a pencil of singular quadratic forms."

(15) Dr. H. R. KINGSTON: "On isothermal nets of curves."

(16) Dr. A. J. KEMPNER: "On polynomials and their residue systems."

(17) Dr. A. L. NELSON: "Note on seminvariants of systems of partial differential equations."

(18) Professor G. A. BLISS: "Solutions of differential equations as functions of the constants of integration."

(19) Professor H. J. ETTLINGER: "On the non-self-adjoint linear system of the second order."

(20) Professor R. D. CARMICHAEL: "Repeated solutions of a certain class of linear functional equations."

The papers of Drs. Wahlin, Forsyth, Chittenden, Kingston, Kempner, and Nelson were read by title. Abstracts, numbered to correspond to the titles in the list above, follow below.

1. In a recent number of the BULLETIN Dr. Wahlin published an article in which he gave a study of the group of residues of the principal units of $\mathfrak{k}(\mathfrak{p}, \mathfrak{a})$ with respect to a certain modulus.

The present paper is a closer study of the principal units along the same lines. The principal units are classified into regular and irregular units, the classification being dependent upon a certain congruence property. Then the smallest number of irregular units that can occur in the base of the group of residues is determined.

2. Perhaps the most useful curve is the Gaussian or probability curve whose equation is $y = y_0 e^{-(x^2/k)}$ or, written in the complete form,

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2},$$

where N is the total area of the curve and σ is the standard deviation, usually defined as the square root of the second unit moment about the mean. However, the curve is ill adapted for one of its most common uses—that of graduation, because the equation is too rigid, having essentially only one parameter. Dr. Forsyth proposes in this paper what amounts to the addition of a second parameter by allowing the exponent of x to be any even integer, the general equation becoming $y = y_0 e^{-x^{2n}/k}$. The values of y_0 and k are then expressed in terms of moments which may be computed for any distribution proposed for graduation. A formula is then derived for determining the most appropriate value of n to be used for graduating a given distribution. A short table of ordinates of the curve for $n = 2$ is added to be used in graduations.

3. Professor Dodd makes a comparison of functions of measurements, postulating that the probability that the error of a measurement will be x is

$$\frac{h}{\sqrt{\pi}} e^{-h^2(x-c)^2}\, dx,$$

where c is the unknown constant error. A comparison with c not zero cannot always be directly inferred from the corresponding comparison with c equal to zero, as was shown previously.* The following extension is, however, immediate: The arithmetic mean is superior to any other weighted mean and to the median, from the standpoint of error-risk, even with c not zero. But if the constant error c is present, then the more numerous the measurements, the more nearly certain it is that the arithmetic mean will incorporate this error c. Thus, in certain cases where an actual "hitting of the mark" is required, the making of measurements beyond a certain number is prejudicial to accuracy; although in the usual case—where "good average results" are desired—the more numerous the measurements, the more satisfactory will be the arithmetic mean, even when a constant error is present.

4. According to Professor E. H. Moore a sequence of functions f_n defined on a range $\mathfrak{P}$ converges relatively uniformly to a limit function f if there exists a scale function σ defined on $\mathfrak{P}$ such that for every small positive number e there exists an integer n_e such that for all $n \geqq n_e$

$$|f_n - f| \leqq e\,|\sigma|$$

uniformly on $\mathfrak{P}$.

* *Monatshefte für Mathematik und Physik*, vol. 24 (1913), pp. 270, 271.

If the functions f_n are continuous on an interval $\mathfrak{P} \equiv (a, b)$, it is shown by Dr. Chittenden that the discontinuities of the limit function f relative to any perfect subset $\mathfrak{P}_0$ of $\mathfrak{P}$ cannot be dense on $\mathfrak{P}_0$ if the convergence is relatively uniform. The condition is also sufficient. From this result it is easy to show that the convergence of Fourier series is not always relatively uniform. Various consequences of this result in relation to the Baire system of classification of functions are considered.

5. Dr. Schweitzer constructs* the complete ordered set of iterative compositions of order k and degree $n + 1$ ($n, k = 1, 2, 3, \cdots$), i. e., iterative compositions of k functions each of which contains not more than $n + 1$ variables. Two cases are distinguished according as the k functions all have the same number of variables or not. In the former case one obtains the complete list of compositions of order $k > 1$ and homogeneous degree, merely by assigning suffixes 1, 2, 3, $\cdots$ in all possible ways to the functional symbols involved in the set of compositions when $k = 1$ and correspondingly affecting by superscripts the arguments of the new functions introduced. In the latter case the required iterative compositions are obtained from the compositions of the former case by increasing or diminishing in all possible ways the number of variables involved, the operation being subject to the notation of the functional symbols of which the variables are arguments.

In the second part of the paper the author constructs "systematically" functional equations corresponding to the preceding iterative compositions by means of the iteration, inversion, elimination, or transformation of variables. A general equation in iterative compositions with all its variables notationally distinct is defined, homogeneous or non-homogeneous with reference to order and degree. Solutions of simultaneous systems are obtained. Examples:

I. $\phi_1\{\phi(x_1, \cdots, x_{n+1}), t_1, \cdots, t_n\} = \phi\{\phi_1(x_1, t_1, \cdots, t_n), \cdots, \phi_1(x_{n+1}, t_1, \cdots, t_n)\}$. For $n = 1$ solutions of this functional equation have been given by Hankel† and the author.

II. $f\{\phi(x, z), f(x, y)\} = \psi(y, z)$, $f_1\{f(x, z), f(x, y)\} = \psi(y, z)$, $f_2\{\phi(x, z), f(x, y)\} = f(y, z)$.

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* Cf. BULLETIN, vol. 23, pp. 301, 302. In Theorem III on p. 302 the specific group of order four is 1, (14)(23), (14), (23).

† Theorie der Complexen Zahlen, p. 34.

III. $\phi\{\phi_1(x_{11}, \cdots, x_{1k}), \cdots, \phi_1(x_{n1}, \cdots, x_{nk})\} = \phi_1\{\phi(x_{11}, \cdots, x_{n1}), \cdots, \phi(x_{1k}, \cdots, x_{nk})\}$ ($k \leqq n$, $n = 2, 3, \cdots$).

IV. $f_1\{f_1(x_{11}, \cdots, x_{1n}), \cdots, f_1(x_{n1}, \cdots, x_{nn})\} = f\{f_1(x_{11}', \cdots, x_{1n}'), \cdots, f_1(x_{n1}', \cdots, x_{nn}')\}$, where x_{ik}' are certain linear homogeneous transformations of x_{ik} with coefficients m_{ik}. In connection with equations of this type it is shown that if

$$f(x_{11}, \cdots, x_{1n}) = \sum_{i=1}^{n} a_{1i}x_{1i}$$

and

$$f_1(x_{11}, \cdots, x_{1n}) = \sum_{i=1}^{n} a_{1i}'x_{1i},$$

the coefficients a_{1t}' are linear homogeneous functions of the coefficients a_{1t} with m's as coefficients. Consequently a group of iterative functional equations of the second order is readily defined.

6. Dr. Schweitzer's definition of mathematics is as follows: mathematics is the systematic construction of clear concepts of real numerical reference, mediate or immediate. The author carefully outlines his meaning, philosophically, of the words in this definition; in particular, his interpretation of "systematic" is very largely Neo-Kantian.

7. Professor Emch's paper deals with the generation of rational circular, in particular, with rational isotropic curves. Every rational curve in a complex z'-plane may be obtained as the transformation of the real axis of a complex z-plane by a rational transformation $z' = f(z)/g(z)$, with coefficients from the domain of ordinary complex (including real) numbers. By such a transformation an algebraic curve is transformed into a curve of the same deficiency, so that rational curves are transformed into rational curves. Necessary and sufficient conditions are established for the polynomials $f(z)$ and $g(z)$, in order that the transformed curves may be circular and, in particular, isotropic. It is shown how these conditions depend upon the theorem: if a polynomial equation $g(z) = 0$, with real coefficients, has imaginary roots only, then $g(z)$ may be represented as the sum of the squares of two polynomials of the same type.

As an application, the generation of all bicircular rational quartics is given explicitly. For example, Bernoulli's lemniscate

$$(x'^2 + y'^2)^2 - 2a^2(x'^2 - y'^2) = 0$$

is represented by

$$z' = \frac{-a\sqrt{2}(z^2 - 1)}{z^2 + 2iz + 1}.$$

8. In this paper, Professor Phillips shows that a cubical determinant of the nth order is the Gibbs multiple product of n dyadics in a space of n dimensions. More generally, the multiple product of m (less than n) such dyadics is a cubical matrix of the nth order with m sheets. As a particular application, it is shown that these cubical matrices can be used instead of determinants in Hedrick's law of analogy which states that a theorem in cubical determinants corresponds to each homogeneous identity in ordinary algebra.

9. If we classify dyadics by the dimensions of their factors, there are four types of dyadics in three dimensions: the one-three and three-one, representing collineations; the one-one and three-three, representing correlations; the two-two, not in general representing a contact transformation; and the one-two, two-three, and their conjugates. Using the symbolic notation introduced by Phillips in a former paper, Professors Phillips and Moore give a treatment of the foregoing types of dyadics and the invariants obtained from them by simple and double multiplication with each other and with idemfactors.
This paper will appear in the *Proceedings of the American Academy of Arts and Sciences.*

10. Special classes of recurrent sequences of integers have been investigated in detail by Lucas and others and numerous interesting theorems have resulted from such study. The purpose of the present paper by Professor Carmichael is to develop certain general properties of the general sequence of integers $u_0, u_1, u_2, \cdots$ defined uniquely in terms of the given initial numbers $u_0, u_1, \cdots, u_{k-1}$ by the recurrence relation

$$u_{x+k} + \alpha_1 u_{x+k-1} + \cdots + \alpha_k u_x = \alpha,$$

in which $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_k$ are given integers. The leading fundamental properties of such sequences are developed by elementary means and several applications of them are given. In particular, many theorems due to Fermat (among them some of his most remarkable discoveries) are readily derived from general results to which one is thus led naturally by elementary means. This association of so many results of Fermat with a single elementary theory tends to suggest the conjecture that the methods of this paper may have been employed by Fermat in some of his investigations.

11. The principal result of Professor Moore's paper is contained in the following theorem: If $f(x)$ is integrable (Lebesgue) in the interval $0 \leqq x \leqq 1$, and in an interval $0 \leqq x \leqq c < 1$ has a first derivative which has a Lebesgue integral and which approaches a finite limit as x approaches $+0$, then the development of $f(x)$ in Bessel's functions of order $\nu(\nu \geqq 0)$ will be convergent to the value $f(+0)$ for $x = 0$, provided $\nu = 0$ or $f(+0) = 0$, and will be uniformly convergent to the value $f(x)$ in the interval $0 \leqq x \leqq c$, provided $\nu = 0$ or $f(+0) = 0$.

12. A set of λ substitutions contained in a given substitution group G is called a set of independent generators of G whenever these λ substitutions generate G, but no $\lambda - 1$ of them have this property. It is clearly always possible to select more than one set of independent generators of G whenever the order of G exceeds 2, and λ is not always the same for the various possible sets of generators of G. In particular, when G is the symmetric group of degree n it is well known that sets can be so chosen that λ may have any value from 2 to $n - 1$, but it does not seem to have been proved that λ can never exceed $n - 1$ when G is a substitution group of degree n. When G is cyclic it is easily seen that λ may have any value from one to the number of different prime factors of the order of G, but it can have no other value.

In the present paper Professor Miller proves the following theorems: If G is any substitution group of degree n with k systems of intransitivity it is always possible to generate G by $n - k$ of its substitutions. When n is written in the form

$$a_0p^m + a_1p^{m-1} + \cdots + a_m,$$

where m is a positive integer and each of the positive integers $a_0, a_1, \cdots, a_m$ is less than the prime number p, then the number of substitutions in each of the possible sets of independent generators in the Sylow subgroup of order p^a contained in G is $a_0m + a_1(m - 1) + \cdots + a_{m-1}$. If one of the transitive constituents of the intransitive subgroup of index p contained in a transitive group G of order p^β is abelian, the number of the independent generators of G cannot exceed one plus the number of the independent generators of this transitive constituent increased by the number of its independent generators whose order exceeds p.

13. In Professor Shaw's paper the general theorems of linear associative algebra are extended from algebras of a finite number of units to algebras with an infinite number of units. These algebras are of two kinds: one having an infinite number of idempotents, the other without idempotents and having no characteristic equations. The latter are called infinitipotent algebras. The most general linear associative algebra will then consist of one infinite or finite semi-simple subalgebra, an infinite or finite nilpotent subalgebra, and an infinitipotent subalgebra. The general theorem of finite algebras is extended to infinite algebras. Some applications are made to recent analysis along the lines of integral equations and integro-differential equations.

14. The reduction of a pencil of singular quadratic forms whose matrix $\lambda A + B$ is singular for all values of λ has been accomplished in a variety of ways. None of the schemes found in the literature enable one to compute the matrix of the reducing substitution with any degree of satisfaction.

Professor Burgess outlines in this note a simple and direct method of accomplishing the reduction, and at the same time computing the matrix of the reducing substitution.

15. In a former paper read before the Society Dr. Kingston discussed some metric properties of nets of plane curves by means of a completely integrable system of three linear partial differential equations of the second order. In particular, the condition for isothermality of the nets was found. In the present paper he exhibits the forms assumed by these equations, when the nets are the various isothermal systems of conics.

16. The following theorem is known: Given any integers

$\alpha_1, \cdots, \alpha_p$, $0 \leqq \alpha_\mu < p$, p prime, then there exist polynomials with integral coefficients such that the α form a complete residue system modulo p of the polynomial. If we demand that $f(\mu) \equiv \alpha_\mu \bmod p$ and that $f(x)$ be of degree less than p, the polynomial is uniquely determined modulo p.*

The results obtained by Dr. Kempner may be stated as follows: Let $m = p_1^{\beta_1} \cdots p_\mu^{\beta_\mu}$, and $\gamma_1, \cdots, \gamma_m$ any integers satisfying $0 \leqq \gamma < m$, then there is usually no polynomial $f(x)$ with integral coefficients for which $\gamma_1, \cdots, \gamma_m$ form a complete residue system modulo m. Necessary and sufficient relations between the γ are developed for the existence of a polynomial of which the γ are a complete residue system (without real loss of generality $f(\mu) \equiv \gamma_\mu \bmod m$ may be assumed). When $\gamma_1, \cdots, \gamma_m$ are of such character that they may form a residue system, there is always a polynomial of degree smaller than $F(m)$, where $F(m)$ is defined as the smallest integer such that $[F(m)]!$ contains m as a factor.

It is next shown how the coefficients of the polynomial may be restricted so that to a possible residue system corresponds exactly one polynomial.

For a given modulus m the number of possible residue systems is determined.

The methods of proof are based on the examination of polynomials $\varphi(x)$ with rational coefficients which are congruent to zero modulo a given m for all $x = 0, \pm 1, \pm 2, \cdots$ in inf. and on the introduction of "arithmetical sequences modulo m of a given order," which are obtained from ordinary arithmetical sequences of a given order by reducing all elements and also all differences modulo m. Many related questions are treated.

17. Wilczynski's method of discussion of the projective differential geometry of a geometrical configuration is based upon either a certain set of ordinary differential equations or a certain completely integrable system of partial differential equations. A necessary step in the discussion is the construction of a fundamental set of seminvariants. In the cases in which the basic system of differential equations has one dependent variable this construction has been accomplished by the reduction of the system to a canonical form, the independent coefficients of which form are the fundamental set

* Zsigmondy, *Monatshefte für Math. und Physik*, vol. 8 (1897), p. 21.

required. If the basic system of differential equations is a completely integrable system of partial differential equations with one dependent variable, Dr. Nelson shows that this process can be made to yield a pseudo-canonical form, the independent coefficients of which are a simpler fundamental set of seminvariants than the usual process gives.

18. The purpose of the paper by Professor Bliss is to prove the differentiability of the solutions of a system of differential equations with respect to the constants of integration by a method which seems more natural and simpler than those which have hitherto been published. Incidentally a restatement of the so-called imbedding theorem for differential equations is given, a theorem which is frequently applied in the calculus of variations, and which has been useful, and could be made still more so, in many other connections. It is analogous to the fundamental theorem for implicit functions in its statement that a solution of a system of differential equations given in advance is always a member of a continuous family of such solutions. The proofs gain much in simplicity by the use of Peano's matrix notations and his notion of the modulus of a matrix.

19. Professor Ettlinger considers the boundary value problem consisting of the differential equation

$$[K(x, \lambda)u_x]_x - G(x, \lambda)u = 0$$

together with the boundary conditions

$$A_i u(a) - A_i' K(a) u_x(a) - B_i u(b) + B_i' K(b) u_x(b) = 0 \qquad (i = 1, 2),$$

which are not self-adjoint. The types of non-self-adjoint conditions are classified and the existence of at least one characteristic number is established for two important cases.

20. The class of functional equations considered by Professor Carmichael in this note includes linear homogeneous differential, difference and q-difference equations. For the general linear homogeneous differential equation

$$a_0 D^n y + a_1 D^{n-1} y + \cdots + a_n y = 0, \tag{1}$$

in which D denotes d/dx, a solution y_1 is said to be r-fold if

$y_1, xy_1, \cdots, x^{r-1}y_1$ are all solutions while $x^r y_1$ is not a solution. If r is greater than unity, the solution is said to be repeated. If y_1 is a repeated solution, then it must also satisfy the equation

$$na_0 D^{n-1}y + (n-1)a_1 D^{n-2}y + \cdots + a_{n-1}y = 0,$$

that is, the equation obtained from (1) by formal differentiation with respect to D. The first elements of the theory of repeated solutions of (1) and a certain more general class of equations thus suggested is developed on a simple postulational basis.

ARNOLD DRESDEN,
Secretary of the Section.

ELEMENTARY INEQUALITIES FOR THE ROOTS OF AN ALGEBRAIC EQUATION.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, October 27, 1917.)

1. LET us write the general algebraic equation in each of the following forms:*

$$x^n = a_1^1 x^{n-1} + a_2^2 x^{n-2} + a_3^3 x^{n-3} + \cdots + a_n^n,$$

$$(1) \qquad x^n = c_{n1}\alpha_1 x^{n-1} + c_{n2}\alpha_2^2 x^{n-2} + \cdots + c_{nn}\alpha_n^n,$$

$$x^n = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n,$$

where

$$a_i^i = c_{ni}\alpha_i^i = \beta_i \qquad (i = 1, 2, \cdots, n),$$

and $c_{n1}, c_{n2}, \cdots, c_{nn}$ denote the binomial coefficients for the power n.

If we let X denote the greatest absolute value of a root of equation (1) and let α denote the greatest absolute value of the quantities $|\alpha_1|, |\alpha_2|, \cdots, |\alpha_n|$, then, as was shown by Carmichael and Mason,† we have $X \geqq \alpha$, the equality sign

* The fruitful and convenient notation employed in the first equation was suggested to me by my friend and colleague, Dr. A. J. Kempner.

† This BULLETIN, vol. 21 (1914), pp. 14–22. Carmichael and Mason stated the theorem for the equation whose roots are the reciprocals of those of (1).