

PROJECTIVE GEOMETRY.

Synthetic Projective Geometry. By DERRICK NORMAN LEHMER.
Boston, 1917. 16mo. xiii+120 pp.

Socrates is a lobster.
Lobsters are mortal; therefore
Socrates is mortal.

It was from some such example as this that we first learned how a correct conclusion could be drawn from a faulty minor premise. The book before us is based upon a similar syllogism.

“The subject of projective geometry is . . . destined soon to force its way into the secondary schools;”

That which is taught in secondary schools needs a clear and very simple text-book; therefore

Projective geometry needs a clear and very simple text-book.

With this conclusion we most heartily agree, we wish that we might also agree with the minor premise. Alas, our ears ring with the cries of those who would abolish geometry from our high schools altogether for the mathematically worded reason that it does not “function.” We suppose that space will continue to exist, even after people have discovered that there is no need to study its properties, and perhaps the nice little book before us will have an archaeological importance long after its subject matter has been proved valueless by the refined tests of laboratory psychology.

It *is* a nice little book, in spite of a bad start and certain other faults which we shall point out in detail. Utterly conventional in type, with the classic methods of the projective geometry of the nineteenth century everywhere in evidence, it is clear, interesting, and readable; the simplest and most elementary book on the subject that we know. The figures are rather small, but the general page impression is pleasing, and the proof-reading seems flawless.

The book, we say, makes a bad start; let us explain. The first chapter deals with one-to-one correspondence, an abstract notion no matter how carefully explained. No later than page 3 we read: “If a one-to-one correspondence has been set up between the objects of one set and the objects of another set, then the inference may usually be drawn that they have the same number of elements.”

“Usually be drawn,” we snort, “and when, pray, may it not be drawn?” We are answered at the bottom of the page.

Two lines AB and $A'B'$ are drawn of different lengths, and put into one-to-one correspondence, “but,” says the author, “it would be absurd to infer from this that there are as many points on AB as on $A'B'$.”

The conclusion that one would naturally draw from this astonishing statement is that the author has never heard of the modern theory of assemblages, but such a conclusion would be erroneous, for we read, page 21: “It is perfectly possible to set up a one-to-one correspondence between the points of a line and those of a plane.”

We receive a second jolt on page 8, where we are told that the totality of points of a line form an infinitude of the first order, while those of a plane form an infinitude of the second order. What is meant is, of course, that the points of a line may be made to depend continuously on one parameter, while those of a plane depend continuously on two parameters. Since, however, it is assumed that the reader's infant mind is not up to understanding such a statement, he is left to puzzle out as best he may how two infinitudes of different orders can be put into one-to-one correspondence. We learn on page 6 that a projective transformation is a continuous transformation and on page 11 we receive this warning: “It must not be forgotten, however, that we are considering only continuous correspondences.” It is a wise restriction; let us turn to page 7. “It is easy to set up a one-to-one correspondence between the points in a plane, and the system of lines cutting across two lines which lie in different planes.” Let us remember that it has never been suggested that a one-to-one correspondence could have exceptional elements, and the plane in question is the projective plane which has the connectivity one. The points on two lines, however, may be put into one-to-one correspondence with the points on a ruled hyperboloid, and the statement is that it is easy to set up a continuous one-to-one correspondence between the points of a surface of connectivity one and one of connectivity two. Finally we have the problem, page 13; remember that this is one of the first problems set to the geometrical infants for whom the book is written.

“Is the axiom ‘The whole is greater than one of its parts’ applicable to infinite assemblages?”

What answer is expected? If the author asks in the mood displayed on page 11, where he says that there is a one-to-one correspondence between the points of a line, and those of a plane, the wise pupil will answer that the axiom has lost all validity. But if the author is in the mood of page 6 where he says that it would be absurd to say that there were the same number of points on two lines of different lengths, then the prudent course would seem to be to reply that the axiom is still doing business at the old stand.

Let us epitomize these ill-natured remarks. We are not suggesting that pupils for whom this book is intended should be taught the theory of assemblages, or the analysis situs. Heaven defend us from any such idea! What we do insist is that if it be bad teaching to tell them things which they do not understand, it is worse teaching to tell them things which are not true. Certainly in their case "It is better not to know so many things, than to know so many things that ain't so."

In the second chapter we get a real start in the proper subject matter of the book. The topics dealt with are the fundamental principles of projection and intersection. Of course Desargues's two triangle theorem is of first importance here; let us see what proof the author gives. He starts ahead in the conventional way, taking the case where the two triangles are in different planes, and this case is done up in good shape. Then the proof comes to an abrupt end* with these words:

"If, now, we consider a plane figure, the points P , Q , and R still lie on a straight line, which proves the theorem."

What the author doubtless means is that if one plane approach the other as a limit, the line P , Q , R will approach a definite limiting position in that plane, so that the proof still holds. What he says, however, is that these three points are collinear, even when the triangles are coplanar. If this be not self-evident, why not prove it, since it is the crux of the situation? But if it be self-evident, why bother with the three-dimensional case in a book on plane geometry?

After this, the book runs ahead smoothly enough for sixteen pages, the next difficulty occurring at that most dangerous point, the fundamental theorem of projective geometry:

"If two projective one-dimensional forms have more than two self-corresponding elements, they are identical."

* P. 16.

This theorem has had a long and stormy career. The present author knows well that it can not be proved by projective synthetic methods without the aid of continuity. He therefore introduces, *ad hoc*, an axiom to the effect that a projective correspondence is a continuous correspondence. Unfortunately, even with the timely assistance of this assumption, the proof is not perfect, for it is assumed* that if at each stage of a certain process the interval which contains a given point is diminished, then the interval itself can be made as small as we please.

The fourth and fifth chapters deal with point rows and linear pencils of the second order, i. e., point and line conics. Pascal and Brianchon play leading rôles, the whole discussion goes ahead smoothly enough. The same may be said of the following chapter which deals with poles and polars, the only exception being a small mistake which is not unknown in elementary text-books of analytic geometry. The polar of a point is defined in a fashion which is inadmissible when the point lies on the conic.† Consequently the following fundamental theorem whereby if one point lie on the polar of a second, the second is on the polar of the first, suffers an exception when one of these points is on the conic. It is better to define the tangent as the polar of a point of the curve; the fundamental theorem then suffers no exception.

The eighth chapter, which might better have been put in the seventh place, deals with the properties of involutions. The author says in the preface that he has never felt satisfied with the usual treatment of involutions by means of circles and anharmonic ratios; the present treatment represents his idea of how the subject may be made easier and more consistent. His treatment is certainly purely projective, and entirely logical, except for the unproved statement‡ that an involution must have two double points or none. It seems to us, however, that the order of topics is artificial, and in consequence, some of the proofs quite needlessly hard. He begins with point involutions; three pairs of collinear points are said to be in involution if they lie on three pairs of opposite sides of a complete quadrangle. Then comes a proof of Desargues's involution theorem for conics which is quite

* P. 32.

† P. 57.

‡ P. 73.

complicated, and lastly what the author calls the fundamental theorem, namely, that if in two projective fundamental one-dimensional forms, a single pair correspond interchangeably, the same is true of every pair. The proof of this covers a whole page, and even then the last details are left to the reader. Then we have dual definitions and theorems for involutions of lines. Finally we deduce from the fundamental theorem that a transversal will meet an involution of concurrent lines in an involution range of points. All this is, as we said, a perfectly legitimate way to handle the subject; we can not think that it is a particularly easy one, for the proofs of these same theorems in such a book as Veblen and Young's "Projective Geometry"* even when written out "in a language understood of the people" are demonstrably shorter than those of the present author.

Chapters VII and IX deal with metrical properties of conics and of involutions respectively. Purists will incline to look upon the introduction of this material as a blemish on the beauty or consistency of the structure. Our own view is just the opposite. These metrical ideas are introduced in illustration of special cases of the theorems developed, not in the course of the logical structure. The chief reason why we welcome them is, however, a didactic one. There can be no doubt that the young geometer feels more interested and more at home when he is dealing with metrical theorems, than when he is occupied with exclusively projective ones. The former certainly come more nearly within his ordinary range of interests. Moreover, the great danger is that projective geometry may appear as a subject apart, but slightly connected with any other mathematical branch. This danger is somewhat obviated when the method of approach is algebraic, as the student sees the connection between the geometric theorems and their algebraic formulation. When, however, as in the present instance, the treatment is purely synthetic, the subject is likely to remain entirely hanging in the air, unless it be tied to the earth by being linked up with more familiar metrical material. We therefore welcome these two chapters, IX being especially interesting and valuable.

The last chapter gives a brief résumé of the history of projective geometry. The author's idea is that the time to learn the history of a subject is after one has found out what the subject is. There is something to be said for this view.

*Vol. 1, Boston, 1910, especially pp. 102, 146.

Now we have exhausted our store of invective. Let us close as we began by saying that it is a clear and interesting little book, whose appearance we heartily welcome.

J. L. COOLIDGE.

NON-EUCLIDEAN GEOMETRY.

The Elements of Non-Euclidean Plane Geometry and Trigonometry. By H. S. CARSLAW. London, Longmans, Green and Company, 1916. 16mo. vii+179 pp.

MORE than a score of years ago a writer of fluent pen published a short article with the title "Is the Non-Euclidean Geometry Inevitable?" His decision was affirmative, his interpretation of the question being "Are the conclusions of non-euclidean geometry inevitable?" We have learnt that they are, but we have also learnt that both the conclusions and the consequences are unavoidable, and that books dealing with the subject must be expected with certainty and met with fortitude. The number already published is already large, as we see by Somerville's compendious bibliography,* and the desire to publish others is so strong that not even the war can choke it. But there is always room at the top, and if a newcomer does not succeed in getting there himself, he raises the existing leader so much the higher. The book before us is a good one. We shall cheerfully damn it in detail later on, let us first praise it in general.

To begin with it is an interesting book. The field covered is not wide. Some things are omitted which we regret to miss, but nothing is included which might better have been left out. The choice of material seems largely guided by the principle that the book was primarily written for teachers, and the author has not shut his eyes to the fact that the average Anglo-Saxon mathematical teacher is too busy to read a long mathematical book, and too weak scientifically to understand a deep one. In consequence of this, certain standard topics like the realization of non-euclidean geometry on surfaces of constant total curvature or the subtleties of the Cayleyan metric are passed over in silence. The most serious omissions come from the restriction to plane geometry. In euclidean

* London, Harrison, 1911.