described show that each such categorical set of postulates will be also "completely independent."

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COMPLETE EXISTENTIAL THEORY OF THE POSTULATES FOR WELL ORDERED SETS.

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A SYSTEM (K, R), where K is a class of elements $A, B, C, \cdots$ and R is a dyadic relation, is called a *well ordered* system when the following conditions are satisfied:*

(*a*) *the system* (K, R) *is a series; and*

(*b*) *every subsystem of* (K, R) *has a leading element.*†

Now when condition (*b*) is added to the conditions (*a*) which define a series, some of the conditions (*a*) become redundant. After eliminating these redundancies, we find the following *three sets of independent postulates for well ordered systems,* each of these three sets being in fact "completely independent" in the sense of E. H. Moore. (The numbering of the postulates is made to conform with that in the preceding note.)

SET I. (POSTULATES 1, 3, 5.)

Postulate 1. $AA \,.=.\, 0.$ (Irreflexiveness.)

Postulate 3. $A \neq B \,.\, AB \,.\, BA :=: 0.$ (Asymmetry for distinct elements.)

Postulate 5. *Every subsystem has at least one leading element.* ("Leadership," or the property of being "supplied with leaders.")

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* G. Cantor, *Math. Annalen*, vol. 49 (1897), p. 208. A. N. Whitehead and B. Russell, Principia Mathematica, vol. 3 (1913), p. 4.

† Here by a *series* we understand any system (K, R) which satisfies any one of the sets of postulates mentioned in the preceding note. A *subsystem* of (K, R) means any system (K', R') such that K' is a subclass of K, and $R' = R$. (Here K' is called a *subclass* of K if every element of K' belongs to K; that is, a subclass is either a part or the whole.) A *leading element* of a system means any element X having the following property: whenever Y is any other element of the system, then $R(XY)$, or simply XY, will be true. (If a system contains only a single element X, then X is a leading element of that system.)

SET II. (POSTULATES 3a, 5.)

Postulate 3a. $AB \,.\, BA :=: 0$.

(Asymmetry for all elements.)

Postulate 5. *Every subsystem has at least one leading element.*

(" Leadership.")

Here 3a implies 1 and 3; and 1 and 3 together imply 3a. Hence set II is equivalent to set I.

SET III. (POSTULATES 1, 5a.)

Postulate 1. $AA \,.=.\, 0$. (Irreflexiveness.)

Postulate 5a. *Every subsystem has just one leading element.*

(" Unique leadership," or the property of being " supplied with unique leaders.")

Here 5a implies 3; for, if AB and BA were both true, and $A \neq B$, the subsystem consisting of the elements A and B would have two leading elements, contrary to 5a. Moreover, 3 and 5 imply 5a; for, if any subsystem had more than one leading element, say X_1 and X_2, then we should have X_1X_2 and X_2X_1, contrary to 3. Hence, set III is equivalent to set I.

It remains to show that every system that satisfies postulates 1, 3, and 5 will satisfy also the missing postulates for serial order, namely 2 and 4:

Postulate 2. $A \neq B :\supset: AB \vee BA$. (Connexity.)

Postulate 4. $A \neq B \,.\, A \neq C \,.\, B \neq C \,.\, AB \,.\, BC :\supset: AC$.

(Transitivity for distinct elements.)

Here 2 follows from 5. For, if neither AB nor BA were true, and $A \neq B$, then the subsystem composed of A and B would have no leading element.

Also, 4 follows from 3 and 5. For, if AC were false, then CA would be true, by 5; but from the truth of AB, BC, and CA, would follow the falsity of BA, CB, and AC, by 3, and hence the subsystem A, B, C would have no leading element.

Thus we see that any one of the sets I, II, III is equivalent to the usual requirement represented by (a) and (b).

Finally, each of the sets I, II, III is " completely independent " in the Moorean sense, as is shown by the following examples, selected from the list used for another purpose in the preceding note.

TABLE II.

Example (K, R).		Character.							
		Set I.			Set II.		Set III.	
No.	Description of R. ($K = 1, 2, 3.$)	1	3	5	3a	5	1	5a
1	12, 13, 23.	.	.	.	.	.	.	.
2	11, 22, 33, 12, 13, 23.	X	.	.	X	.	X	.
5	12, 31, 23.	.	.	X	.	X	.	X
6	11, 22, 33, 12, 31, 23.	X	.	X	X	X	X	X
9	12, 13, 23, 21, 31, 32.	.	X	.	X	.	.	X
10	11, 22, 33, 12, 13, 23, 21, 31, 32.	X	X	.	X	.	X	X
11	12, 21.	.	X	X	X	X	.	X
12	11, 22, 33, 12, 21.	X	X	X	X	X	X	X

An inspection of this table shows that all the types of systems required by Moore's "complete existential theory" for each of the three sets of postulates actually exist. (The entries below the double line in the table are not necessary for the proof.)

In conclusion we note that, by the same device as that used in the preceding note, each of these examples may be enlarged so as to contain n elements ($n > 3$, finite or denumerably infinite), without altering the character of the example. Hence we may readily obtain "categorical" sets of completely independent postulates for every *finite* well ordered system.

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