ON SEPARATED SETS.

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IN the March number of the BULLETIN appeared a discussion of the definition of Lebesgue integrals given in Pierpont's Theory of Functions of Real Variables, volume II, by Fréchet and the author. The questions there discussed are much simplified if use is made of the *outer* and *inner associated sets* of a point set, concepts due, I believe, to W. H. Young.

These sets are defined in the text mentioned, but for the sake of convenience I shall give their definitions here. They arise at once from the definitions of upper and lower measure of a point set. Let A be the set under consideration. Let it be enclosed in an enumerable set R of rectangular cells, of which the sum of the areas is finite and may be denoted by $\bar{R}$. The minimum of all the possible values of $\bar{R}$ is called the upper measure of A and denoted by $\overline{\text{Meas}}\ A$. Now if a sequence $\{R_n\}$ of the rectangular sets is so chosen that $\lim_{n=\infty} \bar{R}_n = \overline{\text{Meas}}\ A$, their divisor (or set of points common to all of them) will contain A, and will be a measurable point set of measure equal to $\overline{\text{Meas}}\ A$ by the ordinary laws of measurable sets. Such a set is called an outer associated set of A and may be denoted by A_e. There will be an infinity of such sets corresponding to a single A, but for each A_e, $\text{Meas}\ A_e = \overline{\text{Meas}}\ A$ and the sets A_e differ only by a set of measure zero. The inner associated sets are defined as follows. Let A be enclosed in a rectangular cell Q, let $B = Q - A$, and let B_e be an outer associated set of B. Let $A_i = Q - B_e$. Then A_i is contained in A, is measurable and $\text{Meas}\ A_i = \text{Meas}\ Q - \text{Meas}\ B_e = \text{Meas}\ Q - \overline{\text{Meas}}\ B = \underline{\text{Meas}}\ A$, by the definition of lower measure. This set A_i is called an inner associated set of A. Young has also shown that any A_i may be regarded as the union of an enumerable set of complete sets C_n contained in A and such that $\lim_{n=\infty} \text{Meas}\ C_n = \underline{\text{Meas}}\ A$. The importance of these sets is obvious;

their existence makes it possible in questions concerning the upper and lower measures of any set A to replace A by a *measurable* set containing A and of measure equal to $\overline{\text{Meas}}\,A$ or contained in A and of measure equal to $\underline{\text{Meas}}\,A$.

Applying these notions to separated sets, we have at once the result that, *if A and B are separated sets, there exist measurable sets A_1 and B_1 enclosing A and B respectively and such that Meas $A_1 = \overline{\text{Meas}}\,A$ and Meas $B_1 = \overline{\text{Meas}}\,B$*; *and further that the measure of the divisor of any such pair, A_1 and B_1, is zero.* The first part of the theorem is obvious; we may take A_e for A_1 and B_e for B_1. To prove the second part, let A_2 and B_2 be measurable sets enclosing A and B respectively and such that the measure of their divisor is zero according to the definition of separated sets. Let A_3 be the divisor of A_1 and A_2, and A_4 the remainder of A_1. Now A_3 contains A, since both A_1 and A_2 do. Therefore Meas $A_3 \geqq \overline{\text{Meas}}\,A =$ Meas A_1. Since A_3 is also $\leqq A_1$, Meas $A_3 =$ Meas A_1 and Meas $A_4 = 0$. Similar results hold for B_3 and B_4, defined in like manner. Thus the divisor of A_1 and B_1 is contained in the divisor of A_2 and B_2, save for at most a set of zero measure made up from A_4 and B_4. Therefore the measure of the divisor of A_1 and B_1 is zero.

The theorem questioned by Fréchet is the following: *Let A and B be separated sets and C their union*; *then $\overline{\text{Meas}}\,C = \overline{\text{Meas}}\,A + \overline{\text{Meas}}\,B$.* The proof can now be given without the use of the ϵ_n-enclosures, which seem to have caused all the trouble. Let A_e, B_e, and C_e be outer associated sets of A, B, and C respectively. Let A_1 be the divisor of A_e and C_e; let B_1 be the divisor ot B_e and C_e; and let C_1 be the union of A_1 and B_1. Also let D be the divisor of A_1 and B_1. Then $A \leqq A_1 \leqq A_e$, $B \leqq B_1 \leqq B_e$ and $C \leqq C_1 \leqq C_e$; hence Meas $A_1 = \overline{\text{Meas}}\,A$, Meas $B_1 = \overline{\text{Meas}}\,B$, Meas $C_1 = \overline{\text{Meas}}\,C$ and by the previous paragraph Meas $D = 0$. Therefore

$$\begin{aligned}\overline{\text{Meas}}\,C &= \text{Meas}\,C_1\\ &= \text{Meas}\,A_1 + \text{Meas}\,B_1 - \text{Meas}\,D\\ &= \overline{\text{Meas}}\,A + \overline{\text{Meas}}\,B - 0,\end{aligned}$$

which was to be proved.

We can also go farther and say that *if the set C is the union* *of A and B, and $\overline{Meas}\ C = \overline{Meas}\ A + \overline{Meas}\ B$, the sets A and B are separated.* Proceeding as in the previous theorem,

$$\begin{aligned}\overline{\text{Meas}}\ C &= \text{Meas}\ C_1 \\ &= \text{Meas}\ A_1 + \text{Meas}\ B_1 - \text{Meas}\ D \\ &= \overline{\text{Meas}}\ A + \overline{\text{Meas}}\ B - \text{Meas}\ D.\end{aligned}$$

But since $\overline{\text{Meas}}\ C = \overline{\text{Meas}}\ A + \overline{\text{Meas}}\ B$, Meas $D = 0$. Thus we have A and B enclosed in measurable sets A_1 and B_1 respectively, of which the divisor D has measure zero. This is the requirement for separated sets.

Regarding the example used by Fréchet and questioned by Pierpont, it can be shown that *any separated partition of a measurable set A will be made up of measurable sets only.*

It is sufficient to prove this for the case that the partition consists of two sets only. For, let B be any one of an enumerable set of separated sets making up A and let U be the union of the remainder. It is readily seen from the definition of separated sets that B and U are separated. Hence it is sufficient to prove B and U measurable.

To do this let B_e and U_e be outer associated sets of B and U respectively, and let B_1 be the divisor of A and B_e, and U_1 the divisor of A and U_e. Then by previous results B_1 and U_1 are measurable, Meas $B_1 = \overline{\text{Meas}}\ B$ and Meas $U_1 = \overline{\text{Meas}}\ U$, and the measure of their divisor is zero.

As A is the union of B_1 and U_1, the set B_1 consists of B and certain points of U contained in B_1. But the divisor of B_1 and U_1 is a null set, hence the divisor of B_1 and U is a null set. Therefore those points of B_1 not belonging to B have measure zero and thus B is the difference between the measurable set B_1 and a null set. Hence B is measurable. In like manner U is measurable. But B was any set of those making up A_1 and so the theorem is proved.

This with the previous theorem gives the important result that no measurable set can be made up of an enumerable set of non-measurable point sets and have the additive property, *i. e.*, $\overline{\text{Meas}}\ A = \overline{\text{Meas}}\ A_1 + \overline{\text{Meas}}\ A_2 + \cdots$, preserved.

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