THE TWENTY–SECOND ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-second annual meeting of the Society was held in New York City on Monday and Tuesday, December 27–28, 1915. The attendance at the four sessions included the following seventy-two members:

Mr. J. W. Alexander II, Professor Clara L. Bacon, Dr. Ida Barney, Professor R. D. Beetle, Mr. D. R. Belcher, Dr. A. A. Bennett, Professor G. D. Birkhoff, Professor E. W. Brown, Dr. T. H. Brown, Dr. R. W. Burgess, Professor F. N. Cole, Professor J. L. Coolidge, Professor D. R. Curtiss, Professor L. E. Dickson, Professor John Eiesland, Professor L. P. Eisenhart, Professor T. C. Esty, Professor F. C. Ferry, Dr. C. A. Fischer, Professor W. B. Fite, Professor Tomlinson Fort, Dr. Meyer Gaba, Mr. W. Van N. Garrettson, Dr. G. M. Green, Professor C. C. Grove, Professor J. G. Hardy, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. Edward Kircher, Professor J. K. Lamond, Dr. D. D. Leib, Dr. P. H. Linehan, Dr. Joseph Lipka, Professor W. R. Longley, Professor C. R. MacInnes, Dr. W. E. Milne, Professor H. H. Mitchell, Professor C. L. E. Moore, Dr. R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor G. D. Olds, Professor W. F. Osgood, Dr. Alexander Pell, Dr. G. A. Pfeiffer, Professor H. B. Phillips, Professor Arthur Ranum, Professor L. H. Rice, Professor R. G. D. Richardson, Dr. P. R. Rider, Mr. J. F. Ritt, Professor J. E. Rowe, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor C. G. Simpson, Professor Mary E. Sinclair, Professor Clara E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Professor Virgil Snyder, Professor Elijah Swift, Mr. H. S. Vandiver, Professor E. E. Whitford, Dr. C. E. Wilder, Professor Ruth G. Wood, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Professor Edward Kasner. The Council announced the election of the following persons to membership in the Society: Professor W. E. Edington, University of New Mexico; Professor J. L. Gibson, University

of Utah; Dr. W. E. Milne, Bowdoin College; Professor L. J. Reed, University of Maine. Nine applications for membership in the Society were received.

The total membership of the Society is now 732, including 73 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 418; the number of papers read was 197. The number of members attending at least one meeting during the year was 253. At the annual election 204 votes were cast. The Treasurer's report shows a balance of $10,470.58, including the life membership fund of $5,560.30. Sales of the Society's publications during the year amounted to $1,832.93. The Library now contains about 5,250 volumes, excluding unbound dissertations.

The Society received with great regret the resignation of Professor L. E. Dickson from the Editorial Committee of the *Transactions*, to take effect October 1, 1916, at the end of fifteen years of editorial services including six years as member of the Editorial Committee. Committees were appointed by the Council to nominate successors to Professor Dickson and Professor D. R. Curtiss, whose first term as member of the Editorial Committee expires on October 1, 1916.

Sixty members and friends attended the annual dinner of the Society on Monday evening.

At the annual election, which closed on Tuesday morning, the following officers and other members of the Council were chosen:

Vice-Presidents,	Professor E. R. HEDRICK,
	Professor VIRGIL SNYDER.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1918,

Professor G. A. BLISS,	Professor W. B. FITE,
Professor R. D. CARMICHAEL,	Professor F. S. WOODS.

The following papers were read at this meeting:

(1) Mr. J. F. RITT: "On the derivatives of a function at a point."

(2) Mr. J. F. RITT: "The finite groups of a class of functions of a real variable."

(3) Professor J. E. ROWE: "A new method of deriving the equation of a rational plane curve from its parametric equations."

(4) Professor H. B. PHILLIPS: "Elastic nets."

(5) Professor ARTHUR RANUM: "The singular points of analytic space curves."

(6) Dr. H. M. SHEFFER: "The reduction of non-monadic relations to monadic" (preliminary communication).

(7) Dr. H. M. SHEFFER: "The elimination of modular existence postulates."

(8) Professor BESSIE I. MILLER: "A new canonical form of the elliptic integral."

(9) Mr. A. R. SCHWEITZER: "On the use of supernumerary indefinables in the construction of axioms."

(10) Professor TOMLINSON FORT: "Linear difference and differential equations."

(11) Dr. DUNHAM JACKSON: "Algebraic properties of self-adjoint systems."

(12) Professor G. D. BIRKHOFF: "On dynamical systems with two degrees of freedom."

(13) Professor G. D. BIRKHOFF: "Infinite products of analytic matrices."

(14) Professor E. B. WILSON: "Ricci's absolute calculus and its application to the theory of surfaces."

(15) Professor C. L. E. MOORE: "Some theorems regarding two-dimensional surfaces in euclidean n-space."

(16) Dr. OLIVE C. HAZLETT: "On the fundamental invariants of nilpotent algebras in a small number of units."

(17) Dr. EDWARD KIRCHER: "Some properties of finite algebras."

(18) Professor M. FRÉCHET: "On Pierpont's definition of integrals."

(19) Professor EDWARD KASNER: "Infinite groups of conformal transformations."

(20) Dr. JOSEPH LIPKA: "Isogonal, natural, and isothermal families of curves on a surface."

(21) Dr. L. L. SILVERMAN: "On the consistency and

equivalence of certain generalized definitions of the limit of a function of a continuous variable."

(22) Professor L. P. EISENHART: " Ruled surfaces generated by the motion of an invariable curve."

(23) Professor L. P. EISENHART: " Transformations of surfaces Ω (second paper)."

(24) Dr. G. M. GREEN: " On rectilinear congruences and nets of curves on a surface."

(25) Professor W. F. OSGOOD: " On infinite regions."

(26) Professor JOHN EIESLAND: " On sphere-flat geometry."

(27) Professor J. L. COOLIDGE: " The meaning of Plücker's numbers for a real curve."

(28) Professor W. M. SMITH: " Characterization of the trajectories described by a particle moving under central force varying inversely as the nth power of its distance from the center of force."

(29) Professor H. H. MITCHELL: " On the generalized Jacobi-Kummer cyclotomic function."

(30) Professor H. H. MITCHELL: " On the congruence $cx^\lambda + 1 \equiv dy^\lambda$ in a Galois field."

(31) Professor R. D. BEETLE: " Sets of properties characteristic of the arithmetic and geometric means."

(32) Dr. R. L. MOORE: " On the foundations of geometry."

(33) Dr. W. C. GRAUSTEIN: " The correspondence of space curves by the transformation of Combescure and by a transformation thereby suggested."

(34) Mr. R. E. GLEASON: " On Dirichlet's principle."

(35) Dr. W. E. MILNE: " On the degree of convergence of Birkhoff's series."

(36) Professor G. C. EVANS: " A generalization of Bôcher's analysis of harmonic functions " (preliminary report).

(37) J. W. ALEXANDER, II: " On the factorization of plane Cremona transformations."

(38) Mr. L. B. ROBINSON: " On elimination between several polynomials in several variables."

Professor Fréchet's paper was communicated to the Society through Professor Curtiss; Mr. Gleason was introduced by Professor Birkhoff. In the absence of the authors Professor Wilson's paper was read by Professor C. L. E. Moore, and the papers of Dr. Sheffer, Professor Miller, Mr. Schweitzer, Professor Fréchet, Dr. Silverman, Professor Coolidge, Dr. Graustein, Professor Evans, Mr. Alexander, and Mr. Robinson were read by title.

Abstracts of the papers follow below.   The abstracts are numbered to correspond to the titles in the list above.

1. Mr. Ritt showed in an earlier paper that, if x is confined to the real domain, the function

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{n!} \left(1 - e^{-\frac{1}{b_n x^2}}\right) \qquad (1 < b_n > |a_n|)$$

has a_n as its nth derivative for $x = 0$, even if

$$\text{lim. sup. } \sqrt[n]{|a_n|/n!} = \infty.$$

This furnished the solution of a problem suggested by Professor Kasner, for which other solutions had been given by Borel and by Serge Bernstein.

In the present paper the domain of the variable is extended over a sector of the complex domain, and a solution of a problem proposed several years ago by Van Vleck is thereby obtained.

By a slight modification of the function above it is shown that a series of negative powers alone can be found which has a_n as its nth derivative for $x = 0$ relative to a sector of the plane.

An extension to the case of several variables is made.

2. In a former paper "On Babbage's functional equation," Mr. Ritt considered the cyclic groups of a class of functions in which the real linear fractional functions are included.

In the present paper it is shown that the only groups of functions of this class are the cyclic and the dihedral.   It is shown also that any two isomorphic groups can be transformed into each other.   The distribution of certain critical points connected with the group is studied.   Applications are made to the theory of real linear transformations.

3. The neatest form of the equation of a rational plane curve of degree n, which we call the R^n, that may be derived from its parametric equations is found by equating to zero the n-rowed determinant obtained by equating to zero the Bezout eliminant of two line sections of the R^n, together with the application of a well-known translation scheme.   The purpose of Professor Rowe's paper is to show how this same

equation may be derived from the equation of the line determined by two points of the R^n. This method involves neither the Bezout eliminant nor the translation scheme.

4. Take a finite number of points in a space of any number of dimensions. Let certain pairs of the points attract each other with a force proportional to the distance. The factors of proportionality for different pairs need not be the same. Let some of the points, called fixed, be held in arbitrarily assigned positions, while the others, called free, adjust themselves in positions of equilibrium. If each pair of points is connected by at least one chain of free points, each attracting the preceding and following, Professor Phillips calls the resulting configuration an elastic net. There is a unique position of equilibrium. The net has some curious properties. For example, if any free point is displaced from its equilibrium position, the force tending to return it is proportional to the displacement and independent of the positions of the fixed points. If any point, free or fixed, is moved, none of the free points can remain at rest, but all move in the same direction distances independent of the positions of the fixed points.

5. In 1901 Burali-Forti classified the singular points of analytic space curves not only with respect to the evenness or oddness of the exponents λ, μ, ν in the expansions

$$x = au^\lambda + \cdots, \qquad y = bu^\mu + \cdots, \qquad z = cu^\nu + \cdots,$$

but also with respect to the values of the curvature $1/r$ and the torsion $1/\rho$ of the curves at the singular points; and by combining the two principles he found fifty classes of such points. His classification, however, has the disadvantage of not being self-dual. In the present note Professor Ranum remedies this defect by taking into account the plane curvature $1/r'$ (defined and discussed by him in a recent number of the *Quarterly Journal*), which is the dual of the point curvature $1/r$. One result is that the number of classes is increased to seventy-four.

6. A postulate set for a deductive system is usually based on one or more undefined classes and one or more undefined relations, each relation being at least dyadic (operations are a special type of relation). By means of the notion of (1, 2)

correspondence Dr. Sheffer shows (1) how to reduce non-monadic relations to monadic; and, thus, (2) how to base postulate sets exclusively on (1, 2) correspondences and classes.

7. Postulate sets for groups, fields, Boolean algebras, and number algebras usually contain one or more existence postulates of the form

(α) There is a K-element x such that $x \circ x = x$,

or of the form

(β) There is a K-element x such that, for every K-element

a, $$a \circ x = x \circ a = a.$$

(Examples: $x =$ the addition modulus, 0; $x =$ the multiplication modulus, 1.) Postulates of type (α) or of type (β) may be called modular existence postulates. Dr. Sheffer shows how, in the construction of postulate sets, modular existence postulates may always be avoided.

8. A particular curve of the family of elliptic norm curves Q_n in S_{n-1} which admit a group G_{2n^2} of collineations is distinguished by a value of the parameter τ, itself an elliptic modular function defined by the modular group congruent to identity (mod n).

In the group G_{2n^2} there are certain involutorial collineations with two fixed spaces. If Q_n is projected from one upon the other, Q_n is mapped by a family of rational curves R_m with the parameter t. Under certain conditions the quadratic irrationality separating involutorial points on Q_n can define the elliptic parameter

$$u = \int \frac{(t dt)}{\sqrt{(t\tau) \alpha_\tau^{n-3} \alpha_t^3}}.$$

In Professor Miller's paper the form u is studied, is contrasted with Klein's form, and its natural occurence in connection with Q_3, Q_4, Q_5 is discussed.

9. In his well-known set of axioms for euclidean geometry Hilbert uses more undefined relations than are necessary for the construction of the desired geometric properties, i. e., on the basis of his system certain of his primitive relations may be defined in terms of others. The possible logical advantage

of such supernumerary indefinables, however, Hilbert does not bring clearly to view. The purpose of Mr. Schweitzer's note is to point out that supernumerary indefinables may sometimes be employed to eliminate explicit reference to existential properties in axioms; this is illustrated by his set of axioms for a field.*

10. Professor Fort's paper is divided into three parts. In the first part difference and differential equations are set up whose coefficients obey certain laws of which periodicity is a very special case and some fundamental theorems are proved for equations of this character.

Part two extends some familiar oscillation theorems primarily due to Sturm to equations of the type considered and proves the continuity of the characteristic values λ_j, for fixed boundary conditions and an interval of fixed length but of variable position, the characteristic values being considered as functions of the initial point of the interval.

Part three considers the so-called self-adjoint boundary value problems for the differential and difference equations where both coefficients depend upon a parameter λ. It is proved that the extremes of a certain type of functions λ_j considered under part two are the characteristic values for the problem in hand. The existence and number of these characteristic values is discussed, and means for distinguishing various cases, etc., taken up. A theorem of oscillation is given.

11. The general definition of adjoint boundary conditions associated with ordinary linear differential equations was given by Birkhoff (*Transactions*, 1908). Bôcher (*Transactions*, 1913) has investigated the circumstances under which a system of the second order is self-adjoint. In Dr. Jackson's paper a condition is given for self-adjoint sets of boundary conditions associated with differential equations of any order. The condition is expressed by a matrix equation of simple form, involving the given coefficients. It becomes particularly symmetric if the given differential equation itself is self-adjoint or anti-self-adjoint.

12. Professor Birkhoff begins by reducing the equations of motion for a dynamical system with two degrees of freedom

* Cf. BULLETIN, vol. 21 (1914–1915), p. 295.

to a special form which he has earlier employed.* In this way he is able to develop simple and general criteria for the existence of periodic orbits of a certain *primary* type.   With the aid of these orbits a reduction of the dynamical problem to a problem in the transformation of surfaces into themselves is effected in a large class of cases.   This reduction affords a means of proving the existence of infinitely many *secondary* periodic orbits, and of determining the precise structure of certain recurrent orbits and of orbits asymptotic to periodic and recurrent orbits.   Finally the structure of the general orbit is determined.   A number of applications of the theorems to special dynamical problems are given.

13. In a large part of the theory of functions of a single complex variable, the matrix of analytic functions rather than the single analytic function must be considered as the fundamental element.   This is certainly the case for the functions defined by linear differential and difference equations.

The goal of the second paper by Professor Birkhoff is to show that the classical theorems of Weierstrass and Mittag-Leffler treating of the formation of functions with singularities of assigned type admit of a natural extension to matrices of analytic functions.

14. Professor Wilson calls attention to Ricci's generally neglected absolute calculus and to its suggestiveness as an implement of research in developing the theory of surfaces of two dimensions in euclidean space of n dimensions.

15. Professor Moore shows that the mean curvature of a two-dimensional surface in higher dimensions than three must be regarded as a vector magnitude and that the properties connected with the curvature of the surface may be expressed very simply with reference to an ellipse in the normal space, the vector from the surface point to the center of the ellipse being the mean curvature.

16. One of the two main outstanding problems in the theory of linear algebras is that of the invariantive classification of nilpotent algebras—that is, algebras such that some power of every number in the algebra is zero.   In this paper Dr.

* *Rendiconti del Circolo Matematico di Palermo*, vol. 39, pp. 265–334.

Hazlett considers rational integral invariants of such algebras. All rational integral invariants for n-ary linear algebras under the total linear group reduce to zero for a nilpotent algebra, and accordingly we consider invariants under the group which leaves unaltered the canonical form. For such invariants this paper proves theorems analogous to theorems about invariants of algebraic forms, and in particular proves the finiteness of the rational integral invariants. The fundamental invariants are found for the simpler cases.

17. Vandiver and Fraenkel have studied finite algebras, but the former did not take up the question of factorization in such an algebra, while Fraenkel in the main restricts himself to decomposable (zerlegbar) rings or algebras. In this paper Dr. Kircher takes up the case of any finite algebra whose elements combine by addition and multiplication subject to the commutative, associative, and distributive laws, the algebra possessing unit elements with regard to both addition and multiplication. Division is not always possible, and when possible is not necessarily unique. Since every algebra fulfilling these conditions can always be represented by the residue classes of a modular system the subject is developed from this point of view. Since the law of unique factorization fails, a partial restoration is effected by the introduction of ideals among which a certain type defined as absolute prime ideals is of great use. These have the same relation to prime ideals in the algebra that an absolute prime modular system has to the irreducible modular system containing it. A number of theorems analogous to well-known number theory theorems are also obtained.

18. Professor Fréchet's paper appears in full in the present issue of the BULLETIN.

19. The groups discussed by Professor Kasner are related to the two types of conformal transformation of order two presented in a paper read at the Providence meeting and there called conformal symmetries and conformal involutions. The first type reverses the orientation of angles while the second preserves it. The groups are infinite, in the sense of involving an infinite number of parameters, and contain subgroups generated by operators of period two.

20. The principal theorem proved in Dr. Lipka's paper is a generalization of the geometric characterization of isogonal trajectories on a surface given by the author in the *Annals of Mathematics*, volume 15, No. 2. The generalized theorem states that if from the geodesic curvature of the ∞^3 geodesic curvature elements composing an isogonal family we subtract the geodesic curvature of the corresponding ∞^3 elements of *any* other isogonal family, and then rotate each element through a right angle, the ∞^3 new elements will form a natural family. This gives a general test for an isogonal family of curves on a surface.

21. In this paper Dr. Silverman establishes a correspondence between certain functions $f(z)$ and certain generalized definitions of the limit of a function of a continuous variable. The results obtained are similar to those presented to the Society in September, 1914, by Silverman and Hurwitz, who established a correspondence between certain functions $f(z)$ and certain definitions of summability of a divergent sequence. As in the case of sequences the following propositions are proved: (1) if $f(z)$ is analytic within and on the boundary of the circle C of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, the corresponding definition is regular, i. e., correctly evaluates any existing limit of a function of a continuous variable; (2) all such definitions are consistent, i. e., if two of them furnish generalized limits of the same function, the values are the same; (3) two definitions are equivalent, i. e., have exactly the same scope of application, provided the corresponding functions have the same zeros with the same multiplicities in C; (4) the definitions for the limit of a function of a continuous variable, corresponding to those of Cesàro and Hölder of the same order for sequences, are equivalent. The last result is not new, having been first proved by Landau in 1913; but it appears here as a special case of (3).

22. One of the outstanding problems in the theory of surfaces is the determination of those surfaces which may be generated in two ways by the motion of invariable curves. The quadrics and surfaces of translation are well-known examples of such surfaces. Professor Eisenhart proposes and solves the problem of finding all ruled surfaces generated by the motion of an invariable curve whose points describe the

generators in the motion.   It is found that cylinders and right conoids are the only surfaces possessing this property.   A right conoid is a surface whose generators meet an axis to which they are perpendicular.   A circular cylinder with the axis of the conoid for an element meets the conoid in a curve which goes into a congruent curve on the surface as the cylinder rolls on the envelope of the family of circular cylinders with the same radii, each having the axis of the conoid for an element.

23. Professor Eisenhart's second paper appeared in full in the January number of the *Transactions*.

24. Starting with any non-conjugate net N on a curved surface S, and a line g passing through each point of the surface and not lying in the tangent plane of that point, Dr. Green associates with the congruence Γ of lines g a second congruence Γ' bearing a certain characteristic relation R to the congruence Γ.   To every point P of S and the line g through it, corresponds a line g' of the congruence Γ' lying in the tangent plane to S at P, and not passing through P.   The relation R between the congruences Γ and Γ' is uniquely determined by the net of parameter curves N; if N is altered, the congruence Γ' is in general also changed.   If N is a conjugate net, the relation R subsists between what Wilczynski* has called the axis and ray congruences.   If N is not conjugate, the generalized axis and ray congruences as defined by Dr. Green in another paper† are also in the relation R.   Probably the most remarkable case in which the relation R exists is that in which the net N is the asymptotic net, and the congruences Γ and Γ' are Wilczynski's directrix congruences. In this connection is given a new geometric characterization of these congruences: if N is asymptotic, then two congruences Γ and Γ' in the relation R have their developables in correspondence if and only if Γ and Γ' are the directrix congruences. Applications of the preceding ideas are made to the general theory of congruences and to the theory of surfaces.

25. Professor Osgood's paper gives a general definition of infinite regions by which ordinary complex n-dimensional space may be closed, projective space, the space of the geometry of

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* *Transactions*, vol. 16 (July, 1915), pp. 311–317.
† Cf. the abstract in the July BULLETIN, vol. 21 (1914–15), pp. 484–5.

inversion, and the space of analysis being the most familiar examples. All such extended spaces are linearly simply connected. An extension of Weierstrass's theorem is then obtained, whereby a function which is meromorphic at every point of such a closed space is rational. It follows that any transformation of such a space into itself, which is regular at every point, is birational, and the Jacobian cannot vanish.

26. Professor Eiesland's paper is a continuation of the author's memoir in the *American Journal of Mathematics*, volume 35, entitled, "On a flat spread-sphere geometry in odd-dimensional space."

In the first part of the paper a method of obtaining surfaces with coordinate lines of curvature is given, based on two theorems which are proved. Surfaces whose lines of curvature are plane in all $n - 2$ systems have been derived and particularly the molding surfaces in S_{n-1} and the generalized Dupin cyclides of the third and fourth order.

The second part of the paper deals with the sphere-flat transformations, and surfaces with coordinate asymptotic lines.

The asymptotic lines on a sphere in $\overline{S}_{n-1}$ are obtained by the integration of a system of differential equations. This integration problem reduces to that of a Riccati equation and quadratures.

27. The usual geometric definition for the Plücker numbers of a plane curve, order, class, etc., is such as to require the recognition of both real and imaginary elements. In Professor Coolidge's paper new definitions are found which are suitable to the case where the universe of discourse includes only the real elements of the plane.

28. Professor Smith's paper shows that the trajectories described by a particle moving under central force varying inversely as the nth power of its distance from the center of force are completely characterized by two properties: (1) That the trajectories be isothermal; (2) That the isoclines be straight lines. By isoclines is meant the curves formed by joining points on consecutive curves which have parallel tangents.

29. If α is a λth root of unity, where λ is any integer, and

q a prime of the form $\lambda\nu + 1$, the Jacobi function $\psi(\alpha)$ has the property that $\psi(\alpha)\psi(\alpha^{-1}) = q$. Professor Mitchell discusses a more general function such that $\psi(\alpha)\psi(\alpha^{-1}) = q^t$, where q is any prime not contained in λ, and t any exponent for which $q^t \equiv 1$, mod. λ. For the case where λ is prime and t is the exponent to which q belongs, mod. λ, this function has been considered by Kummer. The present author determines the ideal factors of the function by essentially the same method which Kummer used in the special case mentioned. An error of Kummer's in this determination is found. Certain properties of these functions are also established.

30. The function $\psi(\alpha)$ considered in Professor Mitchell's first paper bears the same relation to the congruence $cx^\lambda + 1 \equiv dy^\lambda$ for the Galois field of order q^t that the Jacobi function does for the set of integral residues, mod. q, where q is a prime of the form $\lambda\nu + 1$. It is shown in his second paper that the number of solutions of any such congruence is determined if the functions $\psi(\alpha)$ are known. Two applications of this result are made, one to obtain an expression for the number of solutions of such a congruence in a field of order q^{st} in terms of those for a field of order q^t, and another to determine the exact values of these numbers for a field of order q^{2st}, provided $q^t \equiv -1$, mod. λ.

31. If x_1, x_2, $\cdots$, x_n are n observed values, presumably equally accurate, of a magnitude which has an exact, but unknown, value, it is customary to regard the arithmetic mean of the observed values as the most probable value of the magnitude determined by them. A number of writers have attempted to justify this practice by selecting a set of properties which the most probable value ought to possess, and then proving that the set of properties characterizes the arithmetic mean. In his paper, Professor Beetle presents a set of three properties which characterizes the arithmetic mean, and also a set of three properties which characterizes the geometric mean.

If we denote the most probable value determined by x_1, x_2, $\cdots$, x_n by $f_n(x_1, x_2, \cdots, x_n)$, the three properties which characterize the arithmetic mean are

(1) $f_n(x, x, x, \cdots, x) = x$;

(2) $f_n(x_1, x_2, \cdots, x_n)$ is a symmetric function of its arguments;

(3) $f_n(x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$

$$= f_n(x_1, x_2, \cdots, x_n) + f_n(y_1, y_2, \cdots, y_n).$$

The first two of the three properties characteristic of the geometric mean are the same as the first two for the arithmetic mean.   The third is

$(3')$ $f_n(x_1 y_1, x_2 y_2, \cdots, x_n y_n)$

$$= f_n(x_1, x_2, \cdots, x_n) f_n(y_1, y_2, \cdots, y_n).$$

These sets of properties are not only extremely simple, but also exhibit very clearly the essential difference between the arithmetic mean and the geometric mean.   In each set, the three properties are completely independent.

32. Concerning Hilbert's group-theoretic treatment of the foundations of geometry,* Poincaré says (according to Halsted's translation†): " Without doubt this is still not entirely satisfactory since though the *form* of the group is supposed any whatever, its *matter*, that is to say, the plane which undergoes the transformations, is still subjected to being a *number-manifold* in Lie's sense.   Nevertheless, this is a step in advance, ..."

The treatment to which Poincaré here refers is based on three axioms (Axioms I, II and III).   In an abstract‡ of a paper presented to the Society in April, Dr. Moore proposed a set of 12 axioms (Axioms 1–12) for plane analysis situs. These axioms are in terms of *point* and *region*.   He desires to show that if to Axioms 1, 2, 4–7, 9–11 there be added the following Axioms A and B in addition to Hilbert's Axioms I, II and III (interpreted so as to apply properly in this new setting), then every space that satisfies the thus obtained set of axioms is either a euclidean or a Bolyai-Lobachevskian space of two-dimensions according as the group of all motions does or does not contain an invariant subgroup.   This treatment (based on a set of axioms involving the notions point, region, and motion) does not presuppose that space, or even that any part of space, is a number manifold.

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* D. Hilbert, "Ueber die Grundlagen der Geometrie," *Math. Annalen,* vol. 56 (1902).

† Cf. *Science*, May 19, 1911, p. 765.

‡ Cf. this BULLETIN, vol. 21 (July, 1915), pp. 485–486.

A. If O is a point of a region R then there exists a region K containing O and such that K plus its boundary is a subset of R.

B. If A and B are distinct points and every region containing the point O contains a point of the point set M then there exists a point P belonging to M and such that every region containing A and B can be moved into a point set containing O and P.

33. Two curves corresponding point for point so that the tangents at corresponding points are parallel are said to be related by a transformation of Combescure. This transformation finds an analytic parallel in a second point-to-point correspondence of two curves, in which the tangents at corresponding points have as their common perpendicular direction that of the principal normal at the point of the given curve and make with one another an angle whose cosine is equal to the ratio of the corresponding elements of arc, so that the element of arc of the given curve is the projection of that of the transformed curve. The general and special correspondences of these two types are discussed in Dr. Graustein's paper.

34. The object of Mr. Gleason's paper is to establish Dirichlet's principle in n dimensions ($n \geq 2$) under very general conditions by a new method. The method consists partly in the formation of a function $\mathcal{V}$ from a function V, approximately satisfying the given boundary values, by assigning to $\mathcal{V}$, as its value at any point of its region of definition, the weighted mean of V throughout an n-dimensional sphere with center at this point—the sphere being divided into concentric weighted spherical shells.

The n-dimensional region considered is bounded. The (inner) surface distribution v_s on its boundary S is required to be limited and continuous except possibly at a set of points (π) whose projected " area " on the coordinate hyperplanes is of outer content zero. Then (1) if the number of times a point P of S enters as a point of the inner boundary is finite, and either (2) the " projected area " of S on the coordinate hyperplanes is finite or (2') every point is accessible in a certain sense from without, save at a set (π'), there exists a unique function, satisfying the given boundary values save at (π) and (π').

This harmonic function appears as the double limit of any sequences Υ for which the Dirichlet integrals of the corresponding set V, taken over any inner region, approach their lower bound for that region. The method applies also to infinite regions.

35. In the *Transactions*, volume 9, page 373, Professor Birkhoff has shown the general character of the expansion of an arbitrary function $f(x)$ in terms of the characteristic solutions of a certain linear differential system of the nth order, and has proved the convergence of the expansion. In the present paper Dr. Milne studies the degree of convergence of the same expansion, and shows that when $f(x)$ and its first $m - 1$ derivatives vanish at both ends of the interval the remainder after ν terms of the expansion will be less than a constant multiple of $1/\nu^m$ if $f^{(m)}(x)$ is continuous and of limited variation, and less than a constant multiple of $\log \nu/\nu^{m+1}$ if $f^{(m)}(x)$ satisfies a Lipschitz condition.

36. Professor Evans carries out, by means of curvilinear coordinates, a generalization to any curve of a method of Bôcher which was based on the circle. By means of a double integration, regarded as an iterated integration with respect to the curvilinear coordinates, formulas are obtained for solutions of Laplace's and Poisson's equations in terms of the boundary values on an arbitrary curve. The normal derivative of the Green's function thus appears as a ratio of two quantities expressible in terms of the curvilinear coordinates, and therefore can be calculated graphically whenever these can be drawn. Green's theorem is not used in the determination of the expressions for the solutions.

The method possesses the advantage that it applies directly to three dimensions as well as to two.

37. Mr. Alexander gives a simple proof that every Cremona plane transformation is the product of quadratic transformations. The paper will appear in the *Transactions*.

38. In the *American Journal of Mathematics* (1913) Professor Dines published a paper on the " Eliminant of several polynomials." At the suggestion of Professor Coble Mr. Robinson worked out the same problem by a direct method

analogous to the greatest common divisor process. His results were published in the last issue of the Johns Hopkins Circular (July, 1915).

<div align="right">F. N. COLE,
Secretary.</div>

WINTER MEETING OF THE SOCIETY AT COLUMBUS.

THE thirty-sixth regular meeting of the Chicago Section, being the fifth regular meeting of the American Mathematical Society in the west, was held at Columbus, Ohio, on Thursday, Friday and Saturday, December 30, 31, 1915, and January 1, 1916, in affiliation with the American Association for the Advancement of Science.

About one hundred persons were in attendance upon the various sessions, including the following sixty-seven members of the Society: Professor R. B. Allen, Professor Frederick Anderegg, Professor G. N. Armstrong, Professor R. P. Baker, Professor W. H. Bates, Professor P. P. Boyd, Professor Daniel Buchanan, Professor H. T. Burgess, Professor W. D. Cairns, Professor R. D. Carmichael, Professor H. E. Cobb, Professor Elizabeth B. Cowley, Dr. L. C. Cox, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. W. W. Denton, Professor L. E. Dickson, Professor Peter Field, Professor B. F. Finkel, Professor T. M. Focke, Professor W. S. Franklin, Professor Harriet E. Glazier, Professor M. E. Graber, Professor Harris Hancock, Professor E. R. Hedrick, Dr. Cora B. Hennel, Dr. L. A. Hopkins, Professor L. C. Karpinski, Professor A. M. Kenyon, Mr. J. H. Kindle, Professor H. W. Kuhn, Professor Gertrude I. McCain, Dr. J. V. McKelvey, Dr. T. E. Mason, Professor F. E. Miller, Professor G. A. Miller, Professor J. A. Miller, Professor U. G. Mitchell, Professor C. N. Moore, Professor C. C. Morris, Professor F. R. Moulton, Professor A. D. Pitcher, Dr. V. C. Poor, Professor S. E. Rasor, Professor H. W. Reddick, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Professor R. E. Root, Professor D. A. Rothrock, Professor F. H. Safford, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Dr. H. M. Sheffer, Professor H. E. Slaught, Professor K. D. Swartzel, Dr. E. H. Taylor, Mr. C. E. Van Orstrand, Professor C. A. Waldo, Professor C. J. West, Professor H. S. White, Professor E. J. Wilczynski, Professor