Whence

$$f_\alpha g_\beta - f_\beta g_\alpha = \frac{\partial(\phi\psi)}{\partial(\alpha\beta)} f_x g_y + \frac{\partial(\phi\chi)}{\partial(\alpha\beta)} f_x g_z + \frac{\partial(\psi\phi)}{\partial(\alpha\beta)} f_y g_x + \frac{\partial(\psi\chi)}{\partial(\alpha\beta)} f_y g_z.$$

Now if the determinant B is of rank two we must have

$$f_\alpha g_\beta - f_\beta g_\alpha \neq 0, \text{ at the origin;}$$

whence it follows that some one of the functional determinants appearing on the right hand side of the last equality does not vanish at the origin, and hence J is of rank two.

PURDUE UNIVERSITY.

THE HISTORY OF THE CONSTRUCTION OF THE REGULAR POLYGON OF SEVENTEEN SIDES.

Die elementargeometrischen Konstruktionen des regelmässigen Siebzehnecks. Eine historisch-kritische Darstellung. Von R. GOLDENRING. Leipzig und Berlin, Teubner, 1915. 8vo. 6+69 pp. Price 2.80 marks.

TILL near the close of the eighteenth century, mathematicians felt sure that the only regular polygons which could be constructed with ruler and compasses were those known to the Greeks. But the extraordinary discoveries of Gauss, while yet in his teens, greatly extended this class of polygons and settled for all time the limits of possibilities for such constructions. In this connection the discovery that the regular polygon of seventeen sides could be constructed with ruler and compasses was not only one of which Gauss was vastly proud throughout his life, but also, according to Sartorius von Waltershausen,* one which decided him to dedicate his life to the study of mathematics. Two of Gauss's notes recording this turning point of his career have been preserved. The very first entry in his "Wissenschaftliches Tagebuch 1796–1814" † is:

"Principia quibus innititur sectio circuli ac divisibilitas eiusdem geometrica in septemdecim partes etc. *Mart.* 30. *Brunsvigae.*" And again, in his own copy of his Disquisitiones

* Gauss zum Gedächtniss, Leipzig, 1856, p. 16.

† This was "mit Anmerkungen herausgegeben von F. Klein," *Math. Annalen*, Band 57 (1903), pp. 1–34.

Arithmeticae* he wrote the following note† in the margin beside article 365: " Circulum in 17 partes divisibilem esse geometrice, deteximus 1796 Mart. 30."

The first published announcement of this discovery occurred about two months later in the *Intelligenzblatt* of the famous *Allgemeine Literatur-Zeitung*.‡ As files of this periodical are rare§ on this side of the Atlantic, it would seem to be worth while to make more accessible an exact transcription of the announcement. It is as follows:—

" III. Neue Entdeckungen.

Es ist jedem Anfänger der Geometrie bekannt, dass verschiedene ordentliche Vielecke, namentlich das Dreyeck, Viereck, Funfzehneck, und die, welche durch wiederholte Verdoppelung der Seitenzahl eines derselben entstehen, sich geometrisch construiren lassen. So weit war man schon zu Euklids Zeit, und es scheint, man habe sich seitdem allgemein überredet, dass das Gebiet der Elementargeometrie sich nicht weiter erstrecke: wenigstens kenne ich keinen geglückten Versuch, ihre Grenzen auf dieser Seite zu erweiten.

Desto mehr, dünkt mich, verdient die Entdeckung Aufmerksamkeit, dass *ausser jenen ordentlichen Vielecken noch eine Menge anderer, z. B. das Siebenzehneck, einer geometrischen Construction fähig ist.* Diese Entdeckung ist eigentlich nur ein Corollarium einer noch nicht ganz vollendeten Theorie von grösserem Umfange, und sie soll, sobald diese ihre Vollendung erhalten hat, dem Publicum vorgelegt werden.

C. F. Gauss, a. Braunschweig,
Stud. der Mathematik zu Göttingen.

Es verdient angemerkt zu werden, dass Hr. Gauss jetzt in seinem 18ten Jahre steht, und sich hier in Braunschweig mit eben so glücklichem Erfolg der Philosophie und der classischen Litteratur als der höheren Mathematik gewidmet hat.

Den 18 April 96.

E. A. W. Zimmermann, Prof."

Five years later Gauss published the " Theorie von grösserem Umfange " in his Disquisitiones Arithmeticae. The only other reference to the regular polygon of seventeen sides, in

* Lipsiæ, 1801; also in Werke, Bd. 1, Göttingen, 1870.

† *Cf.* Werke, Band 1, p. 476.

‡ Nr. 66, 1 Junius, 1796, col. 554. Two mistakes in this reference are made by Klein, l. c.

§ There is a set in the library of Columbia University.

Gauss's Werke, is in connection with a report of a paper delivered by Erchinger before the Royal Society of Göttingen in 1825.* Gauss gives Erchinger's geometrical construction of the regular 17-side† and remarks that it flows naturally from equations which he had given in the Disquisitiones. He then points out that the merit of Erchinger's paper was not so much in this construction as in the synthetic "proof‡ of its correctness and this is carried through with such admirable, painstaking care to avoid anything not elementary, that it reflects honor on the author and inspires the hope, that his truly uncommon mathematical talent may find every encouragement." While Gauss refers to two of the earlier synthetic constructions of Paucker,§ he remarks that that of Erchinger is "different and carried through more in the spirit of pure geometry."

Goldenring commences his little work by showing that the problem of the solution of the equation $x^{17} = 1$ may be reduced to the solution of certain quadratic equations. Geometrical solutions of such equations are indicated in the next nine pages. Then follow about a score of geometrical constructions for the regular polygon of seventeen sides. They include, of course, the geometrographical construction of Güntsche (1902), the so-called Steinerian constructions of von Staudt (1842) and Schröter (1872), and the Mascheronian construction of Gérard (1897). Two of the constructions given are claimed as new: one by Professor Haussner, of the University of Jena, to whom the book is dedicated, and one by the author himself, through inversion of the von Staudt-Schröter figure. In an appendix (pages 65–66) is given, without any indication of authorship, a construction by means of a right angle.||

* "Geometrische Construction des regelmässigen Siebenzehnecks," *Goettingische Gelehrte Anzeigen*, Dec. 19, 1825, no. 203, p. 2025; Werke, Band 2, pp. 186–187. See also *Bulletin des Sciences mathématiques*, vol. 5 (1826), pp. 299–300.

† Curiously enough, Goldenring gives the impression (pages 15 and 68) that Erchinger's *construction* is unknown.

‡ Unfortunately this proof has not been preserved.

§ (1) "Geometrische Verzeichnung des regelmässigen 17-Ecks und 257-Ecks in d. Kreis," *Jahresverhandl. d. kurländische Gesellschaft für Literatur und Kunst*, Mitau, Band 2, 1822. Apparently unknown to Goldenring. (2) Die ebene Geometrie der geraden Linie und des Kreises. Königsberg, 1823, p. 187. Paucker is also the author of: (3) De divisione geometrica peripheriæ circuli in XVII partes æquales, Königsberg, 1817. This date is incorrectly given as 1814 by Goldenring.

|| This is due to Adler, Theorie der geometrischen Konstruktionen,

Until the publication of Goldenring's pamphlet the most elaborate account of the work done in connection with constructions of the regular 17-side was in the sketch by E. Daniele, "Sulle costruzioni dell' ettadecagono regolare."* While Goldenring has performed a service in presenting something much more elaborate, which is also usefully arranged, it is far from being anything like complete. The only reference to Gauss is to his Disquisitiones and at least two geometrical solutions, published several years before Erchinger's, are nowhere mentioned. These are by John Lowry (1819)† and Samuel Jones (1820).‡ Again, we are informed (page 67) that Ampère's construction announced to the French Academy in 1835§ "does not appear to have been published"; but surely this is the solution "attribuée à Ampère," published, since 1844, in at least five editions of La Frémoire and Catalan's "Théorèmes et problèmes."|| This solution was also given

Leipzig, 1906, p. 227. See also A. Mitzscherling, Das Problem der Kreisteilung, Leipzig, 1913, pp. 73–74.

* Questioni riguardanti le matematiche elementari raccolte e coordinate da F. Enriques. Vol. 2, Bologna, 1914, pp. 167–183.

† *The Mathematical Repository*, new series, vol. 4 (1819), p. 160. Lowry's proof occupies pages 160–168.

‡ The paper dated "Dublin, 17th October, 1819" and read Jan. 24, 1820, was published in *Transactions of the Irish Academy*, vol. 13 (1818), pp. 175–187.

§ "Division de la circonférence de cercle," *Comptes rendus de l'Acad. d. Sc.*, vol. 1 (1835), pp. 119–120. It is here stated that M. Ampère had presented to the academy a geometric figure in which was represented a very simple construction for dividing the circumference of a circle into 17 equal parts. He also announced that he soon expected to read a note the "aim of which was to make clear to those who are still studying elementary geometry, why one can divide, with ruler and compasses, the circumference of a circle into a prime number of equal parts only when this number exceeds unity by a power of 2. M. Ampère will indicate, at the same time, a method leading to this end, that is to say to the desired division in all possible cases and that, without having recourse to any of the theories of higher algebra."

"The utility of introducing these notions into treatises of elementary geometry and the question of why the ancients did not discover the division into 17 parts were questions which led to a discussion by MM. Poinsot, Ampère and Libri. We abstain from giving this discussion at the present time since it has been indicated that it will be renewed at the time when M. Ampère will read the note which he has been content to simply announce to-day."

Apparently this new note was never read. Ampère died in the following year.

|| Théorèmes et problèmes de géométrie élémentaire par H. C. de la Frémoire. Second édition entièrement revue et corrigée par E. Catalan. Paris, 1852, pp. 178–180, 207–209. Sixième éd. 1879, pp. 267–269, 298–302; La Frémoire's name no longer appears on the title page of this edition.

in abridged form by John Casey.*

In the Bibliography (pages 67–69) there are references to 7 pamphlets and 17 special articles on the regular polygon of seventeen sides, and there are some 15 references to more general articles or books in which the same topic is treated. My copy of H. A. Rothe's pamphlet De divisione peripheriæ circuli in 17 et 13 partes æquales was published at Erlangen in 180*5* not 180*4* (page 67).† There are many omissions in the Bibliography. Some of these have been already indicated above. Here are more references which I happen to have met with in the last few years.‡

J. J. Barniville, *Educ. Times Repr.*, vol. 54 (1891), p. 28, question 10176—Bochow, "Eine einfache Berechnung des 17 Ecks," *Zeitschrift für Math. u. Phys.* (Schlömilch), vol. 38 (1893), pp. 250–252—A. Cayley, "On the equation $x^{17}-1=0$," *Messenger of Math.*, vol. 19 (1890) pp. 184–188; Collected Papers, vol. 13 (1897) pp. 60–63—C. H. Chepmell, "In a given circle to inscribe the regular polygon of thirty-four sides," *Educ. Times*, March, 1911; *Educ. Times Repr.* (2), vol. 20 (1911), pp. 51–54—E. Collignon, "Construction du polygone régulier de 17 côtés," *Ass. Franc. Comptes R.*, tome 8 (1879), pp. 162–169—L. Gérard, "Construction du polygone régulier de 17 côtés," *Bull. de Math. élémentaires*, tome 2 (mars, 1897), pp. 164–167—J. A. Grunert, "Reguläre Sieb-

The Kauffmann-Reuschle German translation (Stuttgart, 1858) of the second edition contains (p. 155) the definite statement: "Der hier angeführte geometrische Beweis ist von Ampère."

* For example in his Elements of Euclid, 16th ed., Dublin, 1897, pp. 302–304.

† While there seems to be authority for the statement (p. 67) that H. Birnbaum's paper, "Ueber das reguläre Siebzehneck," was published as a pamphlet in 1834 (e. g., L. A. Sohncke's Bibliotheca Mathematica, Leipzig, 1854, p. 140), there was an edition in 1833 in connection with a school programme, the title page of which contains the invitation: "Zu der öffentlichen Freitags den 29 März 1833 zu haltenden Prüfung der drei obern Classen des Helmstedt-Schöningenschen Gymnasiums und zu dem damit verbundenen Redeacte ladet die Eltern so wie alle Freunde des Schulunterrichts mit geziemender Ehrerbietung ein Dr. Philipp Karl Hess, Professor und Director."

‡ The most extensive bibliography which has been previously published appeared in *L'Intermédiaire des Mathématiciens*, 1897, pp. 23–24, 86, 229; 1899, p. 179; 1901, p. 221; 1902, p. 82 and 1905, p. 112. The contributors were H. Bocard, A. R. Ericsson, E. B. Escott, A. Goulard, Langel and E. Lemoine. Another bibliography, by Max Simon, is given on p. 79 of his Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert, Leipzig, 1906. The astounding inaccuracies throughout the book are here in evidence; for instance: when *Berton* is printed the reader is supposed to know that *Breton* (*de Champ*) was intended, also that *Eninger* stands for *Erchinger*.

zehneck im Kreise," *Archiv. f. Math. u. Phys.* (Grunert), vol. 42 (1864), pp. 361–374—R. Güntsche, "Geometrographische Siebzehnteilung des Kreises," *Archiv f. Math. u. Phys.* (3), vol. 4 (1903)—K. Hagge, "Einfache Behandlung der Siebzehnteilung des Kreises," *Zs. Math. Unterr.*, vol. 41 (1910), pp. 320–325—J. Hoüel, "Sur le polygone régulier de 17 côtés," [Exposition of von Staudt's construction], *Nouv. Annales de Math.*, vol. 16 (1857), pp. 310–311—S. Katayama, "The construction of a regular 17-sided polygon," *The Tôhoku Math. Journal*, vol. 4 (Feb., 1914), pp. 197–202—A. Padoa, "Poligoni regolari di 34 lati. Trattazione elementare," *Boll. di Mat.*, Bologna, vol. 2 (1903), pp. 2–10—W. Schoenborn, Elementare Beweise für einige Gleichungen, die Statt haben zwischen dem Radius eines Kreises, der Seite und der Diagonale der eingeschriebenen regulären 10-, 14-, 18-, 26-, 34-ecke, Pr. Krotoschin, 1873—Steggall, "The value of cos $2\pi/17$ expressed in quadratic radicals," *Proc. Edinb. Math. Soc.*, vol. 7 (1888–89), pp. 4–5.

Of additional references to general articles or books the following may be mentioned: B. Amiot, "Mémoire sur les polygones réguliers," *Nouv. Annales de Math.*, vol. 4 (1844), pp. 264–278—J. W. Butters, "On the solution of the equation $x^p - 1 = 0$ (p being a prime number)," *Proc. Edinb. Math. Soc.*, vol. 7 (1888–89), pp. 10–22—H. S. Carslaw, "Gauss's theorem on the regular polygons which can be constructed by Euclid's methods," *Proc. Edinb. Math. Soc.*, vol. 28 (1910), pp. 121–128—L. E. Dickson, "Constructions with ruler and compasses," in Monographs on Modern Mathematics (1911), 17-side: pp. 371–373—F. Giudice, "Sulla divisione del circolo," *Periodico di mat.* (3), vol. 9 (1912), pp. 161–169—F. Klein, "Elementarmathematik vom höheren Standpunkte aus," Teil I, Leipzig (1908), pp. 122 ff.—K. Kommerell, "Über die Konstruktion der regulären Polygone," *Math. Annalen*, vol. 72 (1912), pp. 588–592—I. L. A. Le Cointe, Leçons sur la théorie des fonctions circulaires et la trigonométrie, Paris (1858), p. 186f.—A. M. Legendre, Traité de trigonométrie at end of Eléments de géométrie, 8ème éd., Paris (1809), § VII, "Du polygone régulier de dix-sept côtés," pp. 419–421—J. Leslie, Elements of geometry, geometrical analysis, and plane trigonometry, second ed., Edinburgh (1811) p. 419 f.*

* It is also of interest to recall the passage in the letter which Sir William Rowan Hamilton wrote to De Morgan in 1852: "Are you *sure* that it is impossible to trisect the angle by Euclid? I have not to lament a single

In conclusion, let us consider approximate constructions of the regular 17-side.* In the seventeenth century C. Renaldini gave an interesting construction for any inscribed regular polygon.† It is as follows: "Construct on the diameter AB of a circumference C, an equilateral triangle ABD; divide AB into n equal parts; join the extremity E, of the second division, to the point D by the secant DEF, then AF is either exactly or approximately the length of the side of the required regular inscribed polygon of n sides." About two centuries later Housel considered the accuracy of this formula for values of n from 3 to 17;‡ for $n = 3$ or 4 or 6 the construction is exact; for $n = 17$ the angle subtended by AF at the center of the circle is about 36′ 37″ too large. Other approximations to the regular 17-side were given by Breton de Champ§ and Catalan‖ and Postula.¶ Catalan pointed out that a closer

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hour thrown away on the attempt, but fancy that it is rather a tact, a feeling, than a proof, which makes us think that the thing cannot be done. No doubt we are influenced by the cubic form of the algebraic equation. But would Gauss's inscription of the regular polygon of seventeen sides have seemed, a century ago, much less an impossible thing, by line and circle?"

De Morgan replied: "As to the trisection of the angle, Gauss's discovery increases my disbelief in its possibility. When $x^{17} - 1$ is separated into quadratic factors, we see how a construction by circles may tell. But, it being granted $ax^3 + bx^2 + cx + d$ is *not* separable into a real quadratic and a linear factor, I cannot imagine how a set of intersections of circles can possibly give no more or less than three distinct points." Graves' Life of Sir Wm. R. Hamilton, vol. 3 (1889), pp. 433–434.

The first rigorous proof of the impossibility of the problem of the trisection of an angle, under euclidean conditions, seems to have been given by L. Wantzel in 1837 (*Liouville*, tome 2, p. 369 f.).

* These are not discussed by Goldenring.

† De resolutione et compositione mathematica, Patavii, 1668, pp. 367–368. Renaldini considered that his construction was accurate in all cases. His error was first shown by Schultz in his Dissertatio de circuli divisione, Königsberg, 1691. (See S. Günther, *Österr. Zeitschrift für Realschulwesen*, vol. 3, pp. 523, 764.) For further history of this problem see A. G. Kästner, Geometrische Abhandlungen, Erste Sammlung, Göttingen, 1790, pp. 266–281; also *Zeits. f. Math. u. Naturw. Unterricht*, vol. 28 (1897), pp. 239, 252–255. Renaldini's approximate construction is sometimes attributed to Bion since it occurs in his Traité de la construction et des principaux usages des instruments de mathématiques, 4e éd., Paris, 1752.

‡ "Division pratique de la circonférence en parties égales," *Nouv. Annales de Math.*, vol. 12 (1853), pp. 77–79. See also remarks on this article by Tempier, *Nouv. Annales de Math.*, vol. 12 (1853), pp. 345–347; vol. 13 (1854), p. 295.

§ *Nouv. Annales de Math.*, vol. 5 (1846), pp. 226–227, 340.

‖ Théorèmes et problèmes de géométrie élémentaire par H. C. de la Frémoire. Second édition entièrement revue et corrigée par E. Catalan, Paris, 1852, pp. 211–212.

¶ E. Catalan, Théorèmes et problèmes de géométrie élémentaire, 6e éd., Paris, 1879, p. 283.

approximation than that of Renaldini is found by taking for the side of the regular inscribed 17-side, one half the difference of the length of a side of the inscribed equilateral triangle and of a side of the regular inscribed hexagon. For the unit circle this leads to the length of a side of the 17-side as 0.36602, which differs from the correct value by about 0.001. According to Renaldini's construction the length is $1/\sqrt{7} = 0.37796\ldots$, which is about 0.02 too large.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.

SHORTER NOTICES.

Plane Trigonometry and Tables. Edited by GEORGE WENTWORTH and DAVID EUGENE SMITH. Ginn and Company, 1915. v + 188 + v + 104 pp. Price, $1.10.

THE formulation of the subject matter in the mind of the teacher largely determines the text he wants to use. This formulation is naturally the product of his experiences with texts studied and taught, and of his own cogitations on the subject and how he can most forcefully and successfully present it. The text under review fits into the plan of the reviewer for present purposes more happily than any of the many texts he has examined.

The authors state that as to sequence of material they have followed the rule "that the practical use of every new feature should be clearly set forth before the abstract theory is developed."

The six functions of acute angles are defined as ratios and put to practical use. The functions of complementary angles, and of angles of 30°, 45°, 60° are developed on pages 7 and 8 and put to immediate use. On page 23 appear the line definitions of the functions. The changes in the functions as the angle changes from 0° to 90° are exhibited through a figure giving the lines representing the six functions. Thus the student is brought to visualize the subject.

The natural trigonometric functions are employed just long enough to be known, and to cause the first twinges of the vexation of multiplication, when the subject of logarithms is