INVARIANTS, SEMINVARIANTS, AND COVARIANTS OF THE TERNARY AND QUATERNARY QUADRATIC FORM MODULO 2.

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1. A SIMPLE and complete theory of seminvariants of a binary form modulo p was given in the writer's second lecture at the Madison Colloquium.* A fundamental system of covariants of a ternary quadratic form F modulo 2 was obtained in the fourth lecture. In place of the method employed there (pages 77–79) to obtain the leading coefficient of a covariant of F, we shall now present a simpler method which makes it practicable to treat also the corresponding question for quaternary quadratic forms. The new method is, moreover, in closer accord with the underlying principle of those lectures, viz., to place the burden of the determination of the modular invariants upon the separation of the ground forms into classes of forms equivalent under linear transformation. By making the utmost use of this principle, we shall obtain a simpler solution of the problem for the ternary case and then treat the new quaternary case.

Let the coefficients of the quadratic form

$$q_n = \Sigma b_i x_i^2 + \Sigma c_{ij} x_i x_j \quad (i, j = 1, \cdots, n;\ j > i)$$

be undetermined integers taken modulo 2. In a covariant of order ω of q_n, the coefficient of x_n^ω is called the leader and also a seminvariant. It is invariant with respect to the group G generated by the linear transformations on $x_1, \cdots, x_{n-1}$ and those replacing x_n by $x_n + l$, where l is a linear function of $x_1, \cdots, x_{n-1}$, the coefficients in each case being integers taken modulo 2.

2. For $n = 2$, G is composed of the transformations

$$x_1 = x_1', \quad x_2 = x_2' + t x_1'.$$

Taking $t = b_1$ and applying the transformation to

* American Mathematical Society Colloquium Lectures, volume IV, New York, 1914; cited later as Lectures.

$$q_2 = b_1x_1^2 + b_2x_2^2 + c_{12}x_1x_2,$$

we get

$$q_2' = \sigma x_1'^2 + b_2x_2'^2 + c_{12}x_1'x_2',$$

where

$$\sigma = b_1(b_2 + c_{12} + 1).$$

Evidently σ, b_2, and c_{12} are seminvariants. Since therefore they completely characterize the classes of forms q_2 under G, they form a fundamental system of seminvariants of q_2. Since c_{12} and

$$J_2 = (b_1 + c_{12} + 1)(b_2 + c_{12} + 1)$$

remain unaltered when b_1 and b_2 are interchanged, they are invariants and, in fact, form a fundamental system of invariants of q_2.

3. For $n = 3$, we have the seminvariants b_3 and

$$P = (c_{13} + 1)(c_{23} + 1),$$

and the invariants (Lectures, pages 69, 74)

$$A = P\,(c_{12} + 1), \quad \Delta = c_{12}c_{13}c_{23} + b_1c_{23} + b_2c_{13} + b_3c_{12},$$
$$J_3 = \beta_1\beta_2\beta_3,$$

of which Δ is the discriminant of q_3, and

$$\beta_1 = b_1 + (c_{12} + 1)(c_{13} + 1), \quad \beta_2 = b_2 + (c_{12} + 1)(c_{23} + 1),$$
$$\beta_3 = b_3 + P.$$

THEOREM. *A fundamental system of seminvariants of q_3 is given by b_3, P, A, Δ, J_3.*

It suffices to prove that they completely characterize the classes of forms q_3 under the group G. We have

$$q_3 = q_2 + lx_3 + b_3x_3^2, \quad l \equiv c_{13}x_1 + c_{23}x_2.$$

(I) $b_3 = P = 0$. Then l is not identically zero and can be transformed within G into x_2. In $q_2 + x_2x_3$ we replace x_3 by $x_3 + c_{12}x_1 + b_2x_2$ and obtain $\Delta x_1^2 + x_2x_3$.

(II) $b_3 = 0$, $P = 1$. Then $l \equiv 0$, $q_3 = q_2$. Thus $A = c_{12} + 1$ and $J_3 = J_2$, which completely characterize the classes q_2 (§ 2).

(III) $b_3 = 1$, $P = 1$. In $q_3 = q_2 + x_3^2$ we replace x_3 by $x_3 + b_1x_1 + b_2x_2$ and obtain $c_{12}x_1x_2 + x_3^2$, where $c_{12} = A + 1$.

(IV) $b_3 = 1$, $P = 0$. As in (I), we may set $l = x_2$. Replacing x_3 by $x_3 + b_1x_1$, we may set also $b_1 = 0$ in q_3. Then $\Delta = c_{12}$, $J_3 = (c_{12}+1)b_2$. Thus, if $\Delta = 0$, $q_2 = J_3x_2^2$. But, if $\Delta = 1$, we replace x_1 by $x_1 + b_2x_2$ and get $q_2 = x_1x_2$. In either case, the coefficients of the final q_3 are determined by the seminvariants.

COROLLARY. *There are exactly eleven linearly independent seminvariants of* q_3; *they may be taken to be* $1, A, \Delta, J_3, AJ_3, b_3, b_3A, b_3\Delta, b_3J_3, P, b_3P$.

In fact, the number of classes in the four cases was 2, 4, 2, 3, respectively. Hence (Lectures, page 13) there are exactly 11 linearly independent seminvariants. The 11 functions in the corollary can be proved to be linearly independent either in the usual direct manner or more simply by noting that any polynomial in b_3, P, A, Δ, J_3 can be reduced modulo 2 to a linear function of the 11 by means of the relations

$$\Delta A = \Delta J_3 = b_3AJ_3 = 0, \quad PA = P,$$

$$P\Delta = b_3(A+P), \quad PJ_3 = (b_3+1)J_3.$$

4. For $n = 4$, the discriminant of q_n is the Pfaffian

$$[1234] = c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23}.$$

Another evident invariant of q_4 is $A_4 = Au$, where

$$u = (c_{14}+1)(c_{24}+1)(c_{34}+1).$$

We shall employ the abbreviations

$$B_1 = b_1 + (c_{12}+1)(c_{13}+1)(c_{14}+1),$$

$$B_2 = b_2 + (c_{12}+1)(c_{23}+1)(c_{24}+1),$$

$$B_3 = b_3 + (c_{13}+1)(c_{23}+1)(c_{34}+1),$$

$$B_4 = b_4 + u.$$

Then* q_4 has the further invariants

$$J_4 = B_1B_2B_3B_4 + b_1b_2b_3b_4[1234],$$

$$K = \Sigma b_1b_2c_{34} + \Sigma b_1(c_{23}c_{24} + c_{23}c_{34} + c_{24}c_{34} + c_{23} + c_{24} + c_{34}) + [1234]\Sigma b_1 + \Sigma c_{12} + \Sigma c_{12}c_{13} + \Sigma c_{12}c_{13}c_{14} + c_{12}c_{13}c_{24}c_{34} + c_{12}c_{14}c_{23}c_{34} + c_{13}c_{14}c_{23}c_{24} + \Sigma c_{12}c_{13}c_{24},$$

* *Proc. Lond. Math. Soc.*, ser. 2, vol. 5 (1907), p. 308.

the subscripts in the final sum being

$$\begin{array}{llll} 12\ 13\ 24, & 12\ 13\ 34, & 12\ 14\ 23, & 12\ 14\ 34, \\ 13\ 14\ 23, & 13\ 14\ 24, & 12\ 24\ 34, & 12\ 23\ 34, \\ 13\ 23\ 24, & 13\ 24\ 34, & 14\ 23\ 24, & 14\ 23\ 34. \end{array}$$

We have the seminvariants b_4 and u since

$$q_4 = q_3 + \lambda x_4 + b_4 x_4^2, \ \lambda = \sum_{i=1}^{3} c_{i4} x_i.$$

THEOREM. *A fundamental system of seminvariants of q_4 is given by b_4, u, [1234], A_4, J_4, K.*

We prove that they characterize the classes q_4 under G.

(I) $b_4 = u = 0$. Then λ is not identically zero and can be transformed within G into x_3. In $q_3 + x_3 x_4$ we replace x_4 by $x_4 + c_{13}x_1 + c_{23}x_2 + b_3 x_3$ and get $x_3 x_4 + q_2$. Then $c_{12} = [1234]$ and $J_2 = K$ characterize the resulting classes (§ 2).

(II) $b_4 = 0$, $u = 1$. Then $\lambda \equiv 0$, $q_4 = q_3$, and

$$A_4 = A, \quad J_4 = J_3, \quad K = \Delta + A + 1$$

form a fundamental system of invariants of q_3 (§ 3).

(III) $b_4 = u = 1$. If $A_4 = 1$, $q_4 = \Sigma b_i x_i^2 + x_4^2$. Replacing x_4 by $x_4 + \Sigma b_i x_i$, we get x_4^2. If $A_4 = 0$, we may take $c_{12} = 1$, replace x_1 by $x_1 + c_{23}x_3$, x_2 by $x_2 + c_{13}x_3$ and have $c_{13} = c_{23} = 0$. Replacing x_4 as before, we get $x_1 x_2 + x_4^2$.

(IV) $b_4 = 1, u = 0$. We may set $\lambda = x_3$ as in (I). Replacing x_4 by $x_4 + c_{13}x_1 + c_{23}x_2$, we have $c_{13} = c_{23} = 0$. Then $[1234] = c_{12}$.

First, let $c_{12} = 0$. Then $K = (b_1 + 1)(b_2 + 1)$, $J_4 = b_3 K$. If $K = 1$, then $b_1 = b_2 = 0, b_3 = J_4$, and q_4 is fixed. If $K = 0$, we may set $b_1 = 1$, replace x_1 by $x_1 + b_2 x_2 + b_3 x_3$ and get $b_1 = 1, b_2 = b_3 = 0$.

Second, let $c_{12} = 1$. Then $J_4 = 0, K = b_1 b_2 + b_3$. Replace x_1 by $x_1 + b_2 x_3$, x_2 by $x_2 + b_1 x_3$, x_4 by $x_4 + b_1 x_1 + b_2 x_2$. We get

$$x_1 x_2 + K x_3^2 + x_3 x_4 + x_4^2.$$

COROLLARY. *There are exactly sixteen linearly independent seminvariants of q_4; they may be taken to be the invariants*

$$1, \ k = [1234], \ A_4, \ J_4, \ K, \ A_4 J_4, \ kK,$$

the products of b_4 by the preceding other than A_4J_4, and

$$u, \quad b_4u, \quad (b_4+1)u\Delta.$$

For, the number of classes in the four cases was 4, 5, 2, 5, respectively. Any polynomial in the six seminvariants given in the theorem can be reduced to a linear function of the sixteen in the corollary by use of*

$$kA_4 = kJ_4 = A_4K = 0, \quad J_4(K+A_4+1) = 0, \quad b_4A_4J_4 = 0,$$

$$uA_4 = A_4, \quad uk=0, \quad uJ_4 = (b_4+1)J_4, \quad uK = (b_4+1)(u+A_4+u\Delta).$$

5. Consider a covariant of odd order ω of q_4

$$C = Sx_4^\omega + S_1x_4^{\omega-1}x_1 + \cdots.$$

Now $x_4 = x_4' + x_1'$ replaces q_4 by q_4' in which

$$(1) \quad c_{12}' = c_{12} + c_{24}, \quad c_{13}' = c_{13} + c_{34}, \quad b_1' = b_1 + b_4 + c_{14}.$$

If the latter replaces S_1 by S_1', we have $S_1' = S_1 + S$. Hence S has no term with the factor $c_{12}c_{13}b_1$. Of the functions in the corollary, only J_4, A_4J_4, and b_4J_4 contain $c_{12}c_{13}b_1$, and its coefficients in them are linearly independent. Hence

$$S = I + b_4I_1 + cu + db_4u + e(b_4+1)u\Delta,$$

where I and I_1 are linear combinations of 1, k, A_4, K, kK.

From geometrical considerations (Lectures, page 72),

$$L = (k+1)\{(B_1+1)x_1 + \cdots + (B_4+1)x_4\}$$

is a covariant. The coefficients of x_4^ω in iL^ω $(i = 1, A_4, K)$ are

$$(k+1)(b_4+1)+u, \quad A_4b_4, \quad (b_4+1)(Kk+K+A_4+u+u\Delta).$$

After subtracting multiples of iL^ω from C we may therefore assume that $c = e = 0$ and that I_1 is free of A_4.

First, let $\omega = 1$. Thus S_1 is derived from S by permuting the subscripts 1 and 4. Then $S_1' = S_1 + S$ gives

$$I = c_{14}I_1 + d(c_{14}+1)\{b_1(c_{24}\alpha + c_{12}c_{34} + c_{34}) + b_4(c_{12}\alpha + c_{13}c_{24} + c_{13})\},$$

where $\alpha = c_{13} + c_{34} + 1$. Let Σ denote the sum of the second member and the function obtained from it by interchanging the subscripts 1 and 2. Thus $\Sigma = 0$. Taking $c_{24} = c_{14}$, we

* The first four from the table of the paper last cited, p. 311.

see that $d = 0$. Thus $(c_{14} + c_{24})I_1 = 0$. Apply $x_2 = x_2' + x_1'$, whence

$$c_{13}' = c_{13} + c_{23}, \quad c_{14}' = c_{14} + c_{24}, \quad b_1' = b_1 + b_2 + c_{12}.$$

Then $c_{14}I_1 = 0$. Hence every $c_{ij}I_1 = 0$, $I_1 = lA_4$. But I_1 is free of A_4. Hence $I_1 = 0$, $I = 0$, $S = 0$.

THEOREM. *Every linear covariant of q_4 is a linear function of L, A_4L, KL.*

Next, let $\omega > 1$. After subtracting from C a constant multiple of $q_4L^{\omega-2}$, whose leader is b_4u, we have $d = 0$ in S. Express S_1 as a polynomial in c_{12}, c_{13}, b_1, and call p the coefficient of their product. The coefficient of $c_{12}c_{13}$ in $S_1' - S_1 = S$, found from (1), is $p(b_4 + c_{14})$, and hence vanishes if $b_4 = c_{14}$; while S itself vanishes if also $c_{24} = c_{34} = 0$. Applying these two conditions to $S = I + b_4I_1$, we find that

$$S = (b_4 + 1)k(n + mK), \qquad n, m \text{ constants}.$$

Several tests failed to exclude this leader. Whether or not there are covariants with such a leader S is not discussed here.

In this connection, note the covariant

$$\Sigma c_{ij}(x_i x_j^{2^r} + x_i^{2^r} x_j) \quad (i, j = 1, \cdots, 4;\ i < j),$$

obtained by replacing the variables in the polar of (x) with respect to q_4 by $x_k^{2^r}$ $(k = 1, \cdots, 4)$.

6. By means of the corollary in § 4, and transformation (1), we readily obtain the

THEOREM. *Every quadratic covariant of q_4 is a linear function of L^2, KL^2, Iq_4, where I is an invariant.*

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THE CONVERSE OF THE HEINE-BOREL THEOREM IN A RIESZ DOMAIN.

BY DR. E. W. CHITTENDEN.

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IN various generalized forms of the Heine-Borel theorem*

* Cf. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo*, vol. 22 (1906), p. 26; and T. H. Hildebrandt, "A contribution to the foundations of Fréchet's calcul fonctionnel," *Amer. Jour. of Mathematics*, vol. 34 (1912), p. 282.