$\frac{2}{3}\sqrt{n}$. This is based upon the fact that two non-commutative substitutions of degree u cannot have less than $\frac{1}{3}u$ letters in common. Professor Manning called attention to the case in which at least one of the substitutions of degree u in the group is of order 2. Here two non-commutative substitutions of degree u have at least $\frac{1}{2}u$ letters in common, from which he concludes that u is greater than $\frac{1}{2}(n - \sqrt{n}) - 1$.

THOMAS BUCK,
Secretary of the Section.

THE SEVENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE seventh regular meeting of the Southwestern Section of the Society was held at the University of Missouri, Columbia, Mo., on Saturday, November 29, 1913. About twenty-five persons attended the meeting, including the following sixteen members of the Society:

Professor L. D. Ames, Professor C. H. Ashton, Dr. Henry Blumberg, Professor W. C. Brenke, Professor E. W. Davis, Dr. E. L. Dodd, Dr. Otto Dunkel, Professor E. R. Hedrick, Professor Louis Ingold, Professor O. D. Kellogg, Dr. A. J. Kempner, Professor W. H. Roever, Professor H. E. Slaught, Professor J. N. Van der Vries, Miss Eula Weeks, Professor W. D. A. Westfall.

The morning session opened at 10.30 A.M. and the afternoon session at 2 P.M. Professors Hedrick and Slaught presided. It was decided to hold the next meeting of the Section at the University of Nebraska on November 28, 1914. The following programme committee was elected: Professor E. W. Davis (chairman), Dr. S. Lefschetz, Professor O. D. Kellogg (secretary). Those present attended a smoker at the house of Professor Kellogg on the evening before the meeting.

The following papers were presented:

(1) Professor W. C. BRENKE: "An example of Abel's integral equation with discontinuous solution."

(2) Professors E. R. HEDRICK and LOUIS INGOLD: "Generalization of Taylor's series."

(3) Dr. S. LEFSCHETZ: "Double integrals of the third kind attached to an algebraic variety."

(4) Professors E. R. HEDRICK and W. D. A. WESTFALL: "Jacobians and existence theorems for implicit functions."

(5) Professor O. D. KELLOGG: "Sign-changes in functions of an orthogonal set."

(6) Dr. E. L. DODD: "The weighting of measurements on the basis of their relative magnitudes."

(7) Dr. A. J. KEMPNER: "On irreducible equations."

(8) Miss EULA WEEKS: "Note on the enclosable property."

(9) Dr. HENRY BLUMBERG: "On an extension of Baire's fundamental theorem concerning functions representable as the limit of a sequence of continuous functions."

(10) Dr. HENRY BLUMBERG: "On the oscillation function and related functions."

(11) Mr. A. R. SCHWEITZER: "On a system of four dimensional simplexes inscribed in a hypersphere."

In the absence of the authors, Dr. Lefschetz's paper was presented by Professor Van der Vries, and Mr. Schweitzer's was read by title. Abstracts of the papers follow.

1. The problem of determining the quantity of flow through a weir notch so that the flow shall be proportional to the depth of liquid in the notch is solved by finding $f(x)$, the form of the notch, from the equation $\int_0^h \sqrt{h - x}\, f(x)dx = kh$, where h is the depth of liquid in the notch. This equation reduces upon differentiation to Abel's integral equation with discontinuous solution. Professor Brenke solves it by means of a substitution of the form $f(x) = \varphi(x) + g(x)$, where $g(x)$ is continuous and $\varphi(x)$ is discontinuous.

2. In this paper, Professors Hedrick and Ingold point out that with a suitably extended notion of orthogonality of functions, developments in Taylor's series may be regarded as expansions in terms of the functions of an orthogonal system, and that many of the usual geometric analogies for such expansions hold also for ordinary Taylor expansions. The notion of orthogonality which is employed involves a set of operations as well as the set of functions under consideration, so that the orthogonality of a set of functions has no meaning until a corresponding set of operations is specified. With this point of view, the difference between a function

and its Taylor development is orthogonal, with respect to certain operations, to all of the functions 1, x, x^2, $\cdots$. This difference, as well as any function orthogonal to the set just given, may well have "Taylor" developments in terms of other functions, $\theta_0, \theta_1, \theta_2, \cdots$, that is, developments in which the coefficients are determined by operations similar to those used in Taylor's series.

3. It has been shown by Picard (Traité des Fonctions algébriques de deux Variables, volume 2, page 231) that there is a minimum ρ to the number of logarithmic singular curves which a simple integral of the third kind belonging to an algebraic surface may have. Dr. Lefschetz has shown the existence of a similar number ρ_v for an algebraic variety. In the present paper he shows that if a double integral of the third kind belonging to an algebraic variety of four dimensions has more than one transcendentally singular surface, it has at least $\rho_v + 1$ of them. It is interesting to note that double integrals, and in all probability multiple integrals, belonging to an algebraic spread do not give any new invariant number corresponding to ρ. It seems likely that Picard's formula for integrals of the second kind (loc. cit., page 409) can be extended with few changes.

4. In this paper Professors Hedrick and Westfall simplify and extend certain theorems in their paper "An existence theorem for implicit functions" read at the summer meeting of the Society. Various forms of difference jacobians are studied, together with their geometric interpretations. Applications are made to inverse transformations.

5. Orthogonality alone of a set of continuous functions does not insure the phenomenon one notices in the sets of orthogonal functions in common use, namely that when arranged in the order of the number of changes of sign, the nth function changes sign at least $n - 1$ times. A sufficient condition for this is the non-vanishing of the determinants $| \varphi_1(x_1), \varphi_2(x_2), \cdots, \varphi_n(x_n) |$ which arise in the interpolation problem. Professor Kellogg's paper calls attention to this fact, shows the condition to be satisfied in the case of a number of the commoner sets of orthogonal functions, and shows as a result that the difference between a function and the approximation

to it by means of the first n terms of its development in terms of a set of orthogonal functions satisfying the above condition, the coefficients being determined after the Fourier manner, changes sign at least $n - 1$ times.

6. In forming a weighted mean of measurements as the most acceptable value for the quantity measured, the weights are usually assigned on the basis of knowledge that some of the measurements are made under more favorable conditions than others. Dr. Dodd discusses the question as to whether the measurements themselves may not be made to yield weights of value in forming weighted means; for instance, should not the median, because of its position among the measurements bear a heavier weight? The "select weighted means" proposed in the paper are compared with the unweighted means from the point of view of reliability.

7. Assuming $\psi(z)$ to be an integral rational function of z with real rational coefficients, irreducible in the natural domain, Dr. Kempner proves a set of theorems on equations $\psi(z) = 0$ having at least one root of rational absolute value, and a corresponding set of theorems on equations $\psi(z) = 0$ having at least one root of rational real part. Although the proofs are of very elementary character, the results are believed to be new. The following theorems represent the type of results obtained. I. *a*. If $\psi(z) = 0$ has a complex root of absolute value 1, $\psi(z)$ is of even order and reciprocal. I. *b*. If $\psi(z) = 0$ has a purely imaginary root, $\psi(z)$ is of even order and does not contain odd powers of z. II. All roots of $\psi(z) = 0$ of rational absolute value (of rational real part) have the same rational absolute value (the same rational real part). III. All equations $\psi(z) = 0$ of degree $n < 10$ having at least one root of rational absolute value (at least one root of rational real part) are completely solvable by radicals. IV. No equation $\psi(z) = 0$ of odd order can have a complex root of rational absolute value (a complex root of rational real part).

8. Referring to Fréchet's letter to Hedrick (*Transactions*, volume 14, page 320), Miss Weeks shows that with the assumptions used by Fréchet, the definition of the enclosable property set up by him is identical with that proposed by Hedrick (*Transactions*, volume 12, page 289), in so far as it is a property of

the fundamental domain. The definitions are identical in the sense that if a set of assemblages of the kind required by Fréchet can be selected, then a set of assemblages satisfying Hedrick's conditions exists. The apparent slight difference in the two statements noticed by Fréchet is therefore only superficial.

9. Baire has shown that a necessary and sufficient condition that a function may be the limit of a sequence of continuous functions is that it be pointwise discontinuous in every perfect set. (Leçons sur les Fonctions discontinues, Paris, 1905, §§ 68, 73, 74 and 77). The proof is given for the case where the domain of the independent variable x is any perfect set. In the *Acta Mathematica*, volume 30, 1906, pages 1 to 48, Baire remarks that the proof applies also to the case where the domain of x is any closed set, and extends the theorem to the case where the domain of x is any set whatever (closed or open). The proof is long, and comparable in difficulty with that of the original theorem. Dr. Blumberg shows how this extension may be made in a very simple and immediate fashion.

10. Let $f(x)$ be any real, bounded, single-valued, continuous or discontinuous function of the real variable x. Let $\omega f(x)$ denote the oscillation of $f(x)$ at the point x, $M(x)$ the maximum at that point, and $m(x)$ the minimum at the point. Dr. Blumberg's paper contains the following results: a necessary and sufficient condition that a function may be an oscillation function; a necessary and sufficient condition that a function may be the oscillation function of an oscillation function; a new proof, by means of the above, of Sierpinski's theorem that $\omega\omega\omega f(x) = \omega\omega f(x)$; the establishment of the inequalities $\omega M(x) \leqq \omega\omega f(x)$ and $\omega m(x) \leqq \omega\omega f(x)$; the proof by means of these inequalities that if $\omega f(x)$ is continuous, both $M(x)$ and $m(x)$ are continuous; the proof that a necessary and sufficient condition that $f(x)$ may be pointwise discontinuous is that $\omega\omega f(x) = \omega f(x)$; a method for finding the complete solution of the equation $\omega f(x) = c(x)$, where $c(x)$ is a given continuous function and $f(x)$ is to be found, with a discussion of more general equations of the same type; indications of extensions of the preceding results to functions of several variables defined in more general domains.

11. In the *American Journal of Mathematics*, volume 34 (1912), page 173, Mr. Schweitzer has shown how to generate a quasi-four-dimensional geometry ${}^4R_4^{(0)}$ by adjoining a point tactically to the system ${}^3R_3$ and assuming the axiom "$\alpha R\beta\gamma\delta\epsilon$ implies $\delta R\epsilon\alpha\beta\gamma$" which ensures that the generating relation is alternating. The resulting system is sufficient for the usual three-dimensional projective geometry if an axiom expressing Dedekind continuity (suitably modified for projective geometry) is added. This geometry ${}^4R_4^{(0)}$ may be regarded as underlying a system of four-dimensional simplexes inscribed in a hypersphere. In the *Archiv der Mathematik und Physik*, volume 21 (1913), page 204, E. Study has remarked that the figure of five ordered real points, no four of which are coplanar, has a (single) property, "signatur" (+) or (−), which is not disturbed by positive real collineations. It seems simpler and altogether more convenient to regard Study's figure of five points with positive or negative "signatur" as a sensed simplex in quasi-four space as indicated above.

O. D. Kellogg,
Secretary of the Section.

THE INFINITE REGIONS OF VARIOUS GEOMETRIES.

BY PROFESSOR MAXIME BÔCHER.

(Read before the American Mathematical Society, September 8, 1913.)

Most geometers are now conscious that the introduction of points at infinity in such a way that in plane geometry they form a line, in three-dimensional geometry a plane, is, to a large extent, an arbitrary convention; but few of them would probably admit that this remark has much practical importance (except in so far as they might regard any question concerning the logical foundation of geometry as having practical importance) since the convention here referred to is commonly regarded as being the only desirable one. It is the object of the present paper to point out more explicitly and in greater detail than has, to my knowledge,