page 78, line 20, for $- H$ read $- H/k(1)$. On page 87, line 12, each denominator Δ should be replaced by $k\Delta$. On page 97, line 8, for $1 + |h'/\rho|$ read $2 + |h'/\rho|$, and make corresponding changes in the succeeding lines. On page 98, line 2, for $- (P' + \epsilon_n)$ read $+ (P' - \epsilon_n)$. On pages 190–197 there is continual confusion of the principal values and their reciprocals.

The general appearance of the page is clear and neat. The functional notation fx instead of $f(x)$ is not at present very widely used, but leads to no confusion here.

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January 14, 1913.

SHORTER NOTICES.

The Teaching of Mathematics in Secondary Schools. By ARTHUR SCHULZE. New York, The Macmillan Company, 1912. 16 mo. xx+367 pp.

IN these piping times when all readers of fifteen-cent magazines, and other patriots, are hastening to climb on the Progressive band wagon, there is grave offense in describing any person or thing as "conservative"; even the anæmic word "moderate" is eyed askance. We do not wish to create an unfavorable opinion of the book before us by attaching to it any of these unpopular predicates; we prefer to call it "eminently sane." The author is an experienced teacher, the difficulties that he faces are those that actually occur in practice, and the ways that he suggests to meet them are sensible and practical. Perhaps the book may be criticized for being a trifle too practical; a little more might be left to the imagination, there is a superabundance in the wealth of detailed illustration which becomes wearisome to the general reader. This is by design, not inadvertence, as the author shows in the preface (page vi) where, in referring to the books of Smith and Young he says: "This book covers a much more restricted field, but does it in greater detail." Perchance he is right. Surely there are a number of teachers who can obtain a good deal more benefit from a chapter on "The equality of triangles" with one hundred twenty-two illustrative examples, than from a comparison of the heuristic method with the individual mode.

The author opens by commenting with pleasing frankness on some of the present shortcomings of our schools; for instance (page 8):

"One would expect the schools to exert a wholesome influence in opposition to this ever growing shallowness. But far from it, they are the worst offenders. . . . There is more taught in many high schools during four years than the average human mind can assimilate in eight." Again (page 13):

"No other subject suffers so much and becomes so valueless as mathematics when treated by mechanical modes of study, and, on the other hand, no other secondary school subject is so admirably adapted to a judicious mode of study as mathematics."

These general considerations lead to the question of why mathematics should claim a place in the crowded secondary school curriculum. This is certainly a live question at present, and the author handles it in admirable fashion. He devotes both Chapters II and XVIII to a judicial balancing of the practical and the disciplinary in mathematical study. He thus replies to those objectors who, clothing themselves in a cloak of mystery called psychology (learnt in one course at summer school), maintain that there is no such thing as disciplining the human mind (page 26):

"If we should accept the theory that the general mental caliber of the student is not improved by study, it would undoubtedly be best to close all the schools after the fourth or fifth year of the grammar school, since the *knowledge* gained afterwards is not worth the trouble."

These considerations lead up to a discussion of the foundations of mathematics. The author takes the generally accepted view that modern researches into foundations have shown the utter futility of attempting to base school geometry upon a set of sufficient, categorical, and independent axioms. In like manner he is sceptical about spending much effort over the fundamental definitions (page 70):

"There exists no flawless definition of a straight line that is fit for school use, and undoubtedly the best policy would be to accept the term without definition."

"Explain an angle as a rotation by using a material contrivance that shows a rotation of a line . . . such illustration will show what an angle really is."

The preliminary chapters close with page 87 and the author enters into a detailed study of the ever debatable subject of plane geometry. An idea of the topics discussed may be obtained from some of the chapter headings: The first propositions in geometry, Original exercises, Equality of triangles, Parallel lines, Limits, Regular polygons.

There is one merit of the author's treatment to which we must call particular attention, his insistence on the importance of a careful analysis of a geometrical problem before undertaking the constructive part of the proof. Here is an example (page 182).

"In equal circles the greater chords subtend the greater (minor) arc.

"*Query*: What is the only means we know to prove the inequality of arcs?

"*Answer*: Unequal central angles.

"*Q*. What therefore must we prove?

"*A*. $\angle O < \angle O'$.

"*Q*. What methods do we know for demonstrating the inequality of angles?" etc.

This scheme of question and answer, when printed at length, bears an unpleasant likeness to the catechism; it is, however, a vital part of geometrical teaching, and has received far too little attention in text-books great and small.

We have so far given the book much praise on didactic grounds; the same might well be continued to the end. Most unfortunately towards the middle the author begins to wabble in his mathematics, and since his work is written for teachers who have a right, if not to the whole truth, at least to nothing but the truth, we must pay some attention to this less attractive aspect of the work. The first difficulty arises in connection with the measurement of the angle between two lines which intersect within or without a circle. The author shows how, if we introduce the idea of positive and negative senses on the circumference, the two usual formulas may be reduced to one; he then continues (page 185):

"If we widen our definition by admitting imaginary arcs, the proposition is true even if one or both sides of the angle do not meet the circumference at all. Thus, if the vertex of the angle moves over the entire plane and its sides rotate in any manner, the proposition always remains true. It does not change abruptly at any point, but is continuous all over the plane. The principle applied here is often referred to as the principle of continuity."

It almost seems as if our author were consciously sinning against the light in writing this. What possible significance can a secondary school pupil attach to the words "imaginary arcs"? Can a teacher who refuses to define a straight line give his class any satisfactory notion of such things? What result can arise from such a process except to teach the pupil to pay himself with empty words? As for the principle of continuity, that has not even the primary merit of being always true. Take the theorem which scandalized the sophists of old: "The sum of two sides of a triangle is greater than the third side." We take an isosceles triangle ABC, where $AB = AC$; A, remaining always on the perpendicular bisector of BC, passes continuously into the imaginary domain and reaches such a position that the altitude $AH = \frac{1}{2}(BC)i$. Now the two equal sides have a length zero, the base is as before.

The author's next lapse occurs a few pages later. We are involved in a discussion of limits and the incommensurable case (page 191).

"We may either tell the student that the theorem can be proved for incommensurable numbers also, but that the proof is too difficult for school work, or we may attempt to make the incommensurable case more plausible by considering approximations of one of these numbers, for instance, the following approximation of $\sqrt{2} = 1.4, 1.41, 1.414, 1.4142$. Obviously the theorem is true for all approximations, hence the two numbers—the numerical measure of the angle and the numerical measure of the arc—can not differ by .1, .01, .001, .0001, etc. Or the error can not be as large as any number, however small, we may assign.

"We have thus proved there can be no *finite* difference between the numerical measure of angle and arc, and this is all the so called rigorous proofs with all their machinery accomplish."

It seems clear from this that the author has an uneasy notion that two constants which do not differ by any "finite" quantity may somehow differ by something else. Has he misunderstood the whole subject of infinitesimals in the calculus and carried away the idea that there are quantities which are less than any assignable quantity, but still not zero?

The discussion of plane geometry is so detailed that we are surprised to find solid geometry polished off in two short chapters. They are well written, especially the discussion of the use of models and the principles for drawing geometrical solids, but many topics of first importance, as the measurement of curved surfaces, the treatment of triedral angles, etc., are omitted. It seems likely that, from this point on, the author felt himself cramped for room, thanks to his early prodigality; for whereas the introductory chapters and the plane geometry cover two hundred sixty-five pages, there are but one hundred pages left for all the rest of the mathematical curriculum. The introductory chapters in algebra are particularly good. The remarks on the choice of material, the placing of emphasis, and the teaching of factoring are excellent. We are less certain as to the didactic wisdom of his advice (page 329) to memorize the formula for solving a quadratic equation. It is far easier to memorize the formula than to understand what is really going on in the solution of a quadratic equation. Let the pupil do each equation at length until he has thoroughly mastered the what and the why, then, perhaps, let him memorize his formula to save time. It is possible also that the author is somewhat over enthusiastic in his praise of graphs; on page 333 we have seven separate reasons for their study, including "The study of graphs enables the student to solve many examples which otherwise he could not solve at all." Doth not the lady protest a little too much?

Unfortunately the algebraic part of the book is marred by mistakes related to those which occur in the geometry. On page 312 is a paragraph headed "The law of no exception." The suggestion of such a precious law at once challenges our interest: we read: "The scientific principle that guides us in such generalizations and that has been called the Law of No Exception or the Principle of Permanence of Equivalent Forms may be stated as follows. In the construction of arithmetic every combination of the previously defined operations (+, −, ×, etc.) shall be invested with a meaning, even when the original definition of the operation excludes such a combination; and the meaning imputed is to be such that the old laws of reckoning still hold good.' "

The credit for this profound statement is attributed to Schubert, and, in fact, we find it on page 14 of his Mathematical Essays and Recreations (Chicago, 1898). There is some obscurity clinging to the letters "etc.," but it seems fair to assume that they include the operation of division, in which case the principle reads: "Good news, we may divide by zero after all."

It is fair to say, that Schubert is entirely willing to take the responsibility for this interpretation, for we read four pages later in his work:

"We discover that, if we apply the ordinary rules of arithmetic to $a \div 0$, all such forms may be equated to one another, both when a is positive and when a is negative. We may, then, invent two new signs for such quotients $+\infty$ and $-\infty$."

We are not sure whether Schubert looks upon the use of this recumbent figure eight as a mathematical recreation. It certainly has no practical utility, it has no connection with the conception of a variable becoming infinite which is so fundamental in the calculus, and it does not come under any law of no exception since the old laws of reckoning do not all apply to it. But Schubert's book is not before us for review, and we prefer to assume that our present author copied this phrase inadvertently. Another inadvertence occurs on page 347:

"To invest $\sqrt{-1}$ and $\sqrt{-4}$ with a meaning, imaginary numbers must be introduced. . . . Imaginary numbers are just as real as other numbers." We do not wish to dispute this if the author will tell us what he means by an imaginary number; is it a real number-pair, a point in the Gauss plane, or merely a graphical symbol? There is no answer given to these questions; the most certain thing which we learn about an imaginary number is that it is real.

We seem to be closing this review with unfriendly comment; that is not the final impression which we wish to give. The faults of the book appear to us in the nature of "removable singularities," its merits are lasting.

J. L. COOLIDGE.

Anharmonic Coördinates. By Lieut.-Colonel HENRY W. L. HIME. Longmans, Green and Company. xiii+127 pp.

THE author's purpose in writing this book was to give a more detailed explanation of anharmonic coordinates than was given by their inventor, Sir W. R. Hamilton. Without laying any claim to originality, he has amplified Hamilton's outline to a degree that makes it quite ready reading as far as method is concerned, though there is a very noticeable amount of algebraic detail that is necessarily abbreviated. The first chapter is devoted to showing how a definite vector may be associated with any given point in the plane by means