furthermore let

$$|f_i(x, t, y)| \leqq M \qquad (i = 1, 2).$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \quad \rho \leqq \frac{\beta}{M}, \quad \rho \leqq \frac{\gamma}{2AM},$$

there exists, in the interval $|x - x_0| \leqq \rho$, *one and only one continuous solution* $y(x)$ *of the integral equation* (1).

The proof is entirely similar to that of Theorem I.

CORNELL UNIVERSITY,
January 10, 1913.

A THEOREM ON ASYMPTOTIC SERIES.

BY MR. VINCENT C. POOR.

THEOREM: *If* $f(z)$ *is not holomorphic at* $z = 0$, *but is formally developable into a Maclaurin series, and if* w *is asymptotic to* $a_1/z + a_2/z^2 + \cdots$ (*written:* $w \backsim a_1/z + a_2/z^2 + \cdots$), *then* $f(w)$ *has an asymptotic representation.**

To prove this theorem take $f(z)$ in the form

$$(1) \qquad \begin{aligned} f(z) &= f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots \\ &\qquad + \frac{f^{(n)}(0)z^n}{n!} + \int_0^z f^{(n+1)}(t) \cdot \frac{(z-t)^n}{n!}\,dt. \end{aligned}$$

Since

$$w \backsim \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

w may be written

$$(2) \qquad w = \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n + \epsilon_{1nz}}{z^n},$$

where, according to the Poincaré definition† for an asymptotic

---

* This theorem is the " résultat préalablement obtenu " referred to in Professor Ford's paper in the *Bulletin* of the French Society for 1911. See *Bulletin Société math. de France*, vol. 40 (1912), fascicule 1 under " Erratum du Tome XXXIX."

† Poincaré, *Acta Mathematica*, vol. 8 (1886), p. 296.

series,

$$\lim_{z \doteq \infty} \epsilon_{1nz} = 0.$$

Replacing z by w in (1) and collecting coefficients of like powers, the following equation results:

$$(3) \qquad f(w) = f(0) + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n + \epsilon_z}{z^n} + R,$$

where

$$R = \int_0^w f^{(n+1)}(t) \cdot \frac{(w-t)^n}{n!}\, dt$$

and where the A_i $(i = 1, \cdots, n)$ are readily determined from the substitution.

$$\epsilon_z = \epsilon_{1nz} + \epsilon_{2nz} + \cdots + \epsilon_{nnz}$$

and

$$\epsilon_{inz}/z^n \qquad\qquad (i = 1, \cdots, n)$$

are the remainders after n terms of $w, w^2, w^3, \cdots, w^n$, respectively. Since

$$\lim_{z \doteq \infty} \epsilon_{inz} = 0 \qquad\qquad (i = 1, \cdots, n),$$

it follows that

$$\lim_{z \doteq \infty} \epsilon_z = 0.$$

To satisfy the Poincaré definition of an asymptotic series, it remains to show that

$$\lim_{z \doteq \infty} z^n R = 0.$$

For z real this last condition may be shown to hold as follows: $(w - t)^n$ does not change sign in the interval $0 \leqq t \leqq w$, therefore the first law of the mean for integrals may be applied. Making this application, it thus obtains that

$$(4) \quad z^n R \equiv \frac{z^n}{n!}\int_0^w f^{(n+1)}(t) \cdot (w-t)^n dt = \frac{z^n \theta_z M}{n!}\int_0^w (w-t)^n dt,$$

where M is the maximum value of $|f^{(n+1)}(t)|$ in the interval $0 \leqq t \leqq w$, and where $-1 \leqq \theta_z \leqq 1$. Evaluating the integral in (4) and taking the limit, it is found that

$$\lim_{z \doteq \infty} z^n R = \lim_{z \doteq \infty} \frac{z^n \cdot \theta_z M \cdot w^{n+1}}{(n+1)!} = 0.$$

Hence by definition

$$f(w) \backsim f(0) + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots. \tag{5}$$

The form I have selected for the remainder in (1) readily applies for z complex. Taking the path of integration as a straight line from $z = 0$ to $z = w$ and writing $t = re^{i\theta}$, $w = w_0 e^{i\theta}$, then

$$|z^n| \cdot |R| \cdot \leqq \frac{|z^n| \cdot M}{n!} \int_0^{w_0} |w_0 - r|^n \cdot dr,$$

M being the maximum value of $|f^{(n+1)}(t)|$ in the interval of integration. But

$$\lim_{z \doteq \infty} |z^n| \cdot |R| \leqq \lim_{z \doteq \infty} \frac{|z^n| M \cdot w_0{}^n}{n!} \int_0^{w_0} dr = \lim_{z \doteq \infty} \frac{z^n M \cdot w_0{}^{n+1}}{n!} = 0.$$

Thus (5) holds for z complex.

That the divergent series

$$f(0) + f'(0) z + \frac{f''(0)}{2!} z^2 + \cdots \tag{6}$$

is asymptotic at $z = 0$, or replacing z by $1/z$ in (6), at $z = \infty$, is evident since w may be chosen as $1/z$ in (5). (6) is therefore an asymptotic representation for $f(z)$ in the vicinity of $z = 0$, or if w be taken as $1/z$ in (5),

$$f\left(\frac{1}{z}\right) \backsim f(0) + \frac{f'(0)}{z} + \frac{f''(0)}{2!\, z^2} + \cdots.$$

ANN ARBOR, MICH.,
January, 1913.